Until now, our analysis had no probabilistic component. Now, we use the structural insights that we gained to bound the probability that $|S(i)|$ and thus $D(i)$ is very large.

Lemma 3.17. *For all $i \in V_n$, it holds that*

$$\mathbf{Pr}(D(i) > 3n) \leq \mathbf{Pr}(|S(i)| > 3n) \leq 2^{-3n}.$$

Proof. For all $e \in P(i)$, let $S(i,e) := \{j \mid j \in S(i), e \in P(j)\}$, i.e. $j \in S(i,e)$ iff it is in $S(i)$ and $P(j)$ contains e. Notice that the $S(i,e)$ are not necessarily pairwise disjoint, but we know that $S(i) = \cup_{i \in P(i)} S(i,e)$ and that implies that $|S(i)| \leq \sum_{e \in P(i)} |S(i,e)|$. By Observation 2.6 and linearity of expectation, $\mathbf{E}[|S(i)|] \leq \sum_{e \in P(i)} \mathbf{E}[|S(i,e)|]$. Now we bound the expected value of $|S(i,e)|$. We do this by using the indicator variables

$$X(i,j,e) = \begin{cases} 1 & \text{if } e \in P(i) \cap P(j) \\ 0 & \text{else} \end{cases}$$

for all $i,j \in V_n$, $i \neq j$, $e \in P(i)$. Observe that

$$|S(i,e)| = \sum_{j \in V_n} X(i,j,e), \text{ thus } \mathbf{E}[|S(i)|] \leq \sum_{e \in P(i)} \mathbf{E}[|S(i,e)|] = \sum_{e \in P(i)} \sum_{j \in V_n} \mathbf{E}[X(i,j,e)].$$

For an edge $e \in P(i)$, let i' be the starting node of e and let i'' be the end node of e. Let r be the index of the bit in which $\texttt{bin}(i')$ and $\texttt{bin}(i'')$ differ. Let

$$\begin{aligned} \texttt{bin}(i') &= x_1 \ldots x_{r-1} x_r x_{r+1} \ldots x_n, && \text{implying that} \\ \texttt{bin}(i'') &= x_1 \ldots x_{r-1} \bar{x_r} x_{r+1} \ldots x_n. \end{aligned}$$

Let $j \in S(i,e)$. Thus, the bit fixing path from j to $\delta(j)$ visits i' and i'' in this order. By the definition of the bit fixing strategy, that means that $\texttt{bin}(j)$ is of the form

$$u_1 \ldots u_{r-1} x_r x_{r+1} \ldots x_n$$

for $u_1, \ldots, u_{r-1} \in \{0,1\}$. Notice that there are at most 2^{r-1} numbers of this form because that is the number of choices for $u_1 \ldots u_{r-1}$. This means that for a given $i \in V_n$ and a given $e \in P(i)$, at most 2^{r-1} indicator variables $X(i,j,e)$ can be one, all other are zero because j will never be routed through i', no matter what $\delta(j)$ is. Let $J(i,e)$ be this set of the 2^{r-1} bad starting locations that can possibly be routed through e.

The destination $\delta(j)$ is chosen uniformly at random from V_n. Even if $j \in K(i,e)$, the bit fixing path to $\delta(i)$ only visits i' and i'' if $\delta(j)$ is of the form

$$x_1 \ldots x_{r-1} \bar{x_r} z_{r+1} \ldots z_n$$

for $z_{r+1}, \ldots, z_n \in \{0,1\}$. Since the first r bits are fixed, there are 2^{n-r} numbers of this form. The probability that one of them is chosen out of the 2^n possible values for $\delta(j)$ is $2^{n-r}/2^n = 1/2^r$ because $\delta(j)$ is chosen uniformly at random from V_n. Thus, for $j \in J(i,e)$, the probability for $X(i,j,e) = 1$ is $1/2^r$. We get that

$$\mathbf{E}[|S(i)|] \leq \sum_{e \in P(i)} \mathbf{E}[|S(i,e)|] = \sum_{e \in P(i)} \sum_{j \in V_n} \mathbf{E}[X(i,j,e)] = \sum_{e \in P(i)} \sum_{j \in J(i,e)} \mathbf{E}[X(i,j,e)]$$

$$= \sum_{e \in P(i)} |J(i,e)| \cdot \frac{1}{2^r} = n \cdot 2^{r-1} \cdot \frac{1}{2^r} = \frac{n}{2}$$

because $P(i)$ has at most n edges. We now know that $n/2$ is an upper bound on $\mathbf{E}[|S(i)|]$ and thus $3n \geq 6\mathbf{E}[|S(i)|]$. Furthermore, $|S(i)|$ is a sum of Bernoulli random variables which means that we can apply Theorem 3.9. It yields that

$$\mathbf{Pr}(|S(i)| > 3n) \leq 2^{-3n}.$$

Applying Lemma 3.16 concludes the proof. □

Notice that Lemma 3.15 and Lemma 3.16 have purely deterministic arguments. However, for the deterministic routing strategy that just uses bit fixing paths without intermediate locations, we saw an example where a lot of packages are routed through the same vertex, and, ultimately, over the same edges. This means that $S(i)$ is large for a lot of vertices.

Furthermore, the lemmata follow in the same way if $P(i)$ is the bit fixing path from from $\delta(i)$ to $\pi(i)$ in phase 2 and $D(i)$ is the delay of a_i in phase 2. Lemma 3.17 uses that the destination is chosen uniformly at random. However, the argument works analogously for phase 2, where the starting point is chosen uniformly at random. The following lemma thus can be proven similarly to Lemma 3.17.

Lemma 3.18. *Let $P'(i)$ be the bit fixing path from $\delta(i)$ to $\pi(i)$. Let $T'(i)$ be the number of steps that a_i needs to travel from $\delta(i)$ to $\pi(i)$ and define $D'(i) = T'(i) - |P'(i)|$. For all $i \in V_n$, it holds that*

$$\mathbf{Pr}(D'(i) > 3n) \leq 2^{-3n}.$$

Now the proof of the following theorem is nearly finished.

Theorem 3.14. *For any $\pi : V_n \to V_n$, Valiant's randomized routing strategy has an execution time of at most $8n$ steps with probability $1 - \frac{1}{2^n}$.*

Proof. The probability that $D(i)$ is more than $3n$ is bounded by 2^{3n}, thus the probability that a_i arrives after more than $4n$ time steps is bounded by 2^{3n} as well since $P(i)$ has at most n edges. By the Union Bound, the probability that there is at least one package that takes more than $4n$ steps is bounded by $2^n \cdot 2^{-3n} = 2^{-2n}$. Thus, phase 1 is finished after $4n$ steps with probability $1 - 2^{-2n}$, and the same holds for phase 2. The probability that both phases finish after $4n$ steps is thus at least

$$(1 - 2^{-2n})(1 - 2^{-2n}) = 1 - 2 \cdot 2^{-2n} + 2^{-4n} \geq 1 - 2 \cdot 2^{-2n} \geq 1 - 2^{-n}.$$

□