Cosntruction of AVD

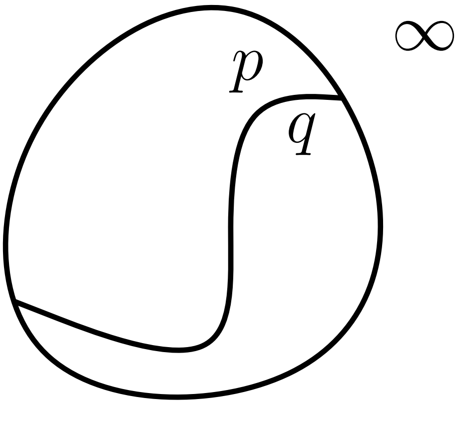
Rolf Klein, Kurt Mehlhorn, Stefan Meiser, "Randomized Incremental Construction of Abstract Voronoi Diagrams," Computational Geometry, vol 3., no. 3, pp.157–184, 1993.

Finite Part of AVD

- Let Γ be a simple closed curve such that all intersections between bisectring curve lie inside the inner domain of Γ
- Consider a site ∞, define $J(p,\infty) = J(\infty,p)$ to be Γ for all sites $p \in S$, and $D(\infty,p)$ to be the outer domain of Γ for all sites $p \in S$.

Incremental Construction

- Let $s_1, s_2, \ldots, s_n$ be a random squence of S
- Let R_i be $\{\infty, s_1, s_2, \ldots, s_i\}$
- Iteratively construct $V(R_2), V(R_3), \ldots, V(R_n)$



General Position Assumption

- No $J(p,q)$, $J(p,r)$ and $J(p,t)$ intersect the same point for any four distinct sites, $p, q, r, t \in S$
  $\rightarrow$ Degree of a Voronoi vertex is 3

Remark

- For $1 \leq i \leq n$ and for all sites $p \in R_i$, $\mathrm{VR}(p, R_i)$ is simply connected, i.e., path connected and no hole
- If $J(p,q)$ and $J(p,r)$ intersect at a point x, $J(q,r)$ must pass through x

Basic Operations

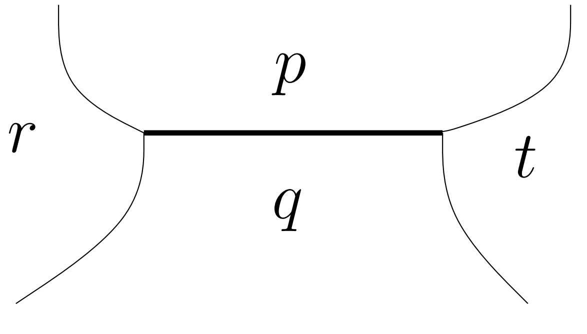
- Given $J(p,q)$ and a point v, determine $v \in D(p,q)$, $v \in J(p,q)$, or $v \in D(q,p)$
- Given a point v in common to three bisecting curves, determine the clockwise order of the curves around v
- Given points $u \in J(p,q)$ and $w \in J(p,r)$ and orientation of these curves , determine the first point of $J(p,r)\mid_{(w,\infty]}$ crossed by $J(p,q)|_{(v,\infty]}$
- Given $J(p,q)$ with an orientation and points v, w, x on $J(p,q)$, determine if v come before w on $J(p,q)\mid_{(x,\infty]}$

Notation: Give a connected subset A of R^2, int$A$, bd$A$, and cl$A$ mean the interior, the boundary, and the closure of A, respectively.

Conflict Graph $G(R)$, where R is R_i for $2 \leq i \leq n$

- bipartitle graph (U, V, E)
- U: Voronoi edges of $V(R)$
- V: Sites in $S \setminus R$
- $E : \{(e,s) \mid e \in V(R), s \in S \setminus R, e \cap \mathrm{VR}(s, R \cup \{s\}) \neq \emptyset\}$
  - a conflict relation beteween e and s.

Remark:
a Voronoi edge is defined by 4 sites under the general position assumption



Lemma 1

Let $R \subseteq S$ and $t \in S \setminus R$. Let e be the Voronoi edge between $\mathrm{VR}(p, R)$ and $\mathrm{VR}(q, R)$. $e \cap \mathrm{VR}(t, R \cup \{t\}) = e \cap \mathrm{R}(t, \{p, q, t\})$. (Local Test is enough)

Proof:

$\subseteq$: Immediately from $\mathrm{VR}(t, R \cup \{t\}) \subseteq \mathrm{VR}(t, \{p, q, t\})$

$\supseteq$: Let $x \in e \cap \mathrm{VR}(t, \{p, q, t\})$

- Since $x \in e$, $x \in \mathrm{VR}(p, R) \cup \mathrm{VR}(q, R)$ and $x \notin \mathrm{VR}(r, R) \supseteq \mathrm{VR}(r, R \cup \{t\})$ for any $r \in R \setminus \{p, q\}$.
- Since $x \in \mathrm{VR}(t, \{p, q, t\})$, $x \notin \mathrm{VR}(p, \{p, q, t\}) \cup \mathrm{VR}(q, \{p, q, t\}) \supseteq \mathrm{VR}(p, R \cup \{t\}) \cup \mathrm{VR}(q, R \cup \{t\})$
- $x \notin \mathrm{VR}(r, R \cup \{t\})$ for any site $r \in R \rightarrow x \in \mathrm{VR}(t, R \cup \{t\})$

Insertiong $s \in S \setminus R$ to compute $V(R \cup \{s\})$ and $G(R \cup \{s\})$ from $V(R)$ and $G(R)$. Handle a conflict between s and a Voronoi edge e of $\mathrm{VR}(R)$

Lemma 2

$\mathrm{cl}\ e \cap \mathrm{cl}\ \mathrm{VR}(s, R \cup \{s\}) \neq \emptyset$ implies $e \cap \mathrm{VR}(s, R \cup \{s\}) = \emptyset$

proof

- Let x belong to $\mathrm{cl}\ e \cap \mathrm{cl}\ \mathrm{VR}(s, R \cup \{s\})$
- x is an endpoint of e:
  - x is the intersection among three curves in R
  - For any $r \in R$, $J(s, r)$ cannot pass through x due to the general position assumption
  - $x \in D(s, r) \rightarrow$ the neighborhood of $x \in D(s, r)$
  - $\exists y \in e$ belongs to $\mathrm{VR}(s, R \cup \{s\})$
- $x \in e \cap \mathrm{bd}\ \mathrm{VR}(s, R \cup \{s\})$
  - $x \in J(p, q) \cap J(s, r)$
  - a point $y \in e$ in the neighborhood of x such that $y \in \mathrm{VR}(s, R \cup \{s\})$

Let $\mathcal{Q}$ be $\mathrm{VR}(s, R \cup \{s\})$

Lemma 3

$\mathcal{Q} = \emptyset$ if and only if $\deg_{G(R)}(s) = 0$

proof $(\rightarrow)$ If $\mathcal{Q} = \emptyset$, $\deg_{G(R)}(s) = 0$

$(\leftarrow)$

- $\deg_{G(R)}(s) = 0$ implies cl $\mathcal{Q} \subseteq$ int $\mathrm{VR}(r, R)$ for some $r \in R$
- $\mathrm{VR}(r, R \cup \{s\}) = \mathrm{VR}(r, R) - \mathcal{Q}$
- Since $\mathrm{VR}(r, R \cup \{s\})$ must be simply connected, $\mathcal{Q} = \emptyset$

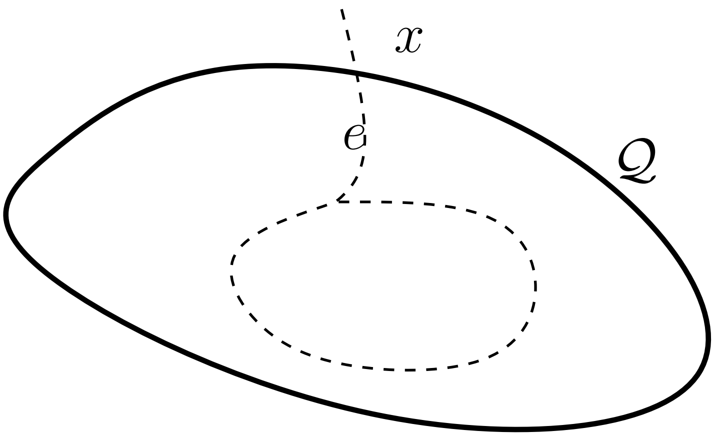
Lemma 4

Let I be $V(R) \cap \mathrm{cl}\ \mathcal{Q}$.

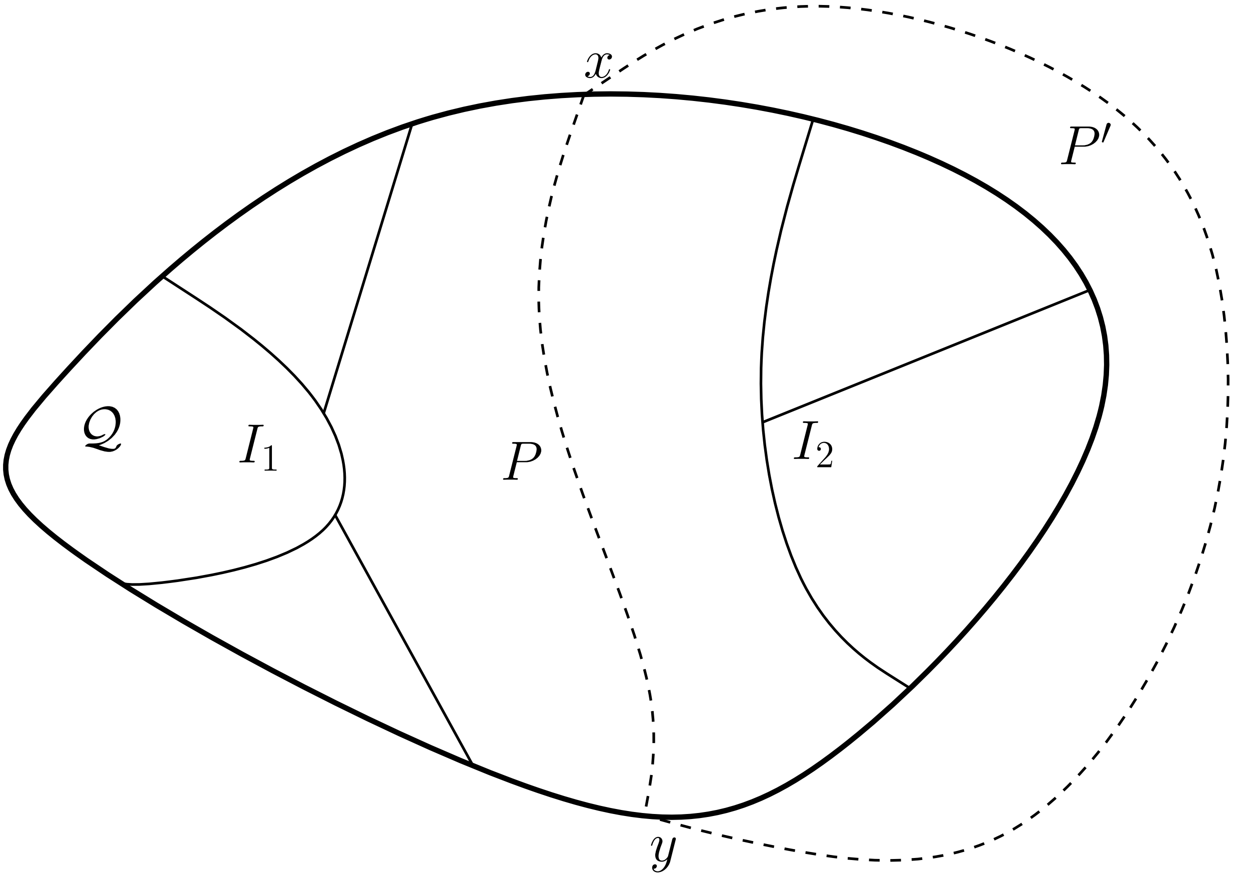
I is a connected set which intersects bd $\mathcal{Q}$ in at least two points.

Proof:

- bd $\mathcal{Q}$ is a closed curve which does not go through any vertex of $V(R)$ due to the general position assumption.
- Let $I_1, I_2, \ldots, I_k$ be connected components of I
- Claim: I_j, $1 \leq j \leq k$, contains two points of bd $\mathcal{Q}$.
  - If I_j contains no point, $I_j \subseteq$ int $\mathcal{Q}$. In other words, for some $r \in R$, $\mathrm{VR}(r, R)$ contains I_j, contradicting that $\mathrm{VR}(r, R)$ must be simply connected
  - If I_j intersects exactly one point x on bd $\mathcal{Q}$, let e be the Voronoi edge of $V(R)$ which contains x. Then both sides of e belong to the same Voronoi region. There exists a contradiction.



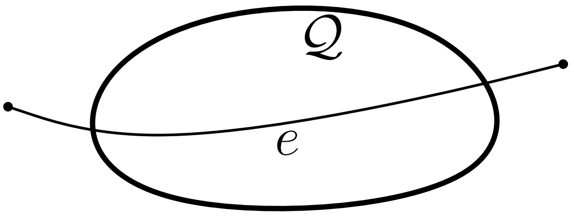
- Assume the contrary that $k \geq 2$
  - There is a path $P \subseteq \text{cl } \mathcal{Q} - (\cup_{1 \leq j \leq k} I_j)$ connects two points on bd $\mathcal{Q}$ such that one component of $\mathcal{Q} - P$ contains I_1 and the other component contains I_2.
  - Let x, y be the two endpoints of P and let $r \in R$ such that $P \subseteq \text{VR}(r, R)$.
  - Since $x, y \notin V(R)$, $\text{VR}(r, R \cup \{s\}) = \text{VR}(r, R) - \mathcal{Q} \neq \emptyset \rightarrow x, y \in \text{cl VR}(r, R \cup \{s\})$
  - Since $x, y \in \text{cl VR}(r, R \cup \{s\})$, there is a path $P' \subseteq \text{VR}(r, R \cup \{s\})$ with endpoints x and y.
  - $P \circ P'$ is contained in cl $\text{VR}(r, R)$ and contains either I_1 and I_2, contradicting cl $\text{VR}(r, R)$ is simply connected



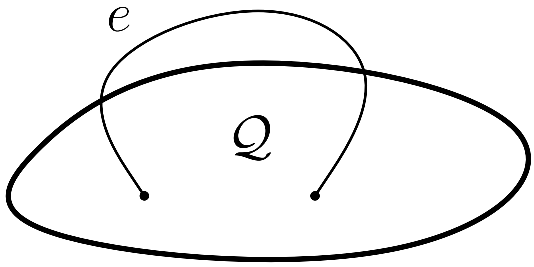
Lemma 5

Let e be an edge of $V(R)$. If $e \cap \mathcal{Q} \neq \emptyset$,

- either ($e \cap \mathcal{Q} = V(R) \cap \mathcal{Q}$ or $e \cap \mathcal{Q}$ is a single component),
- or $e - \mathcal{Q}$ is a single component



$e \cap \mathcal{Q}$



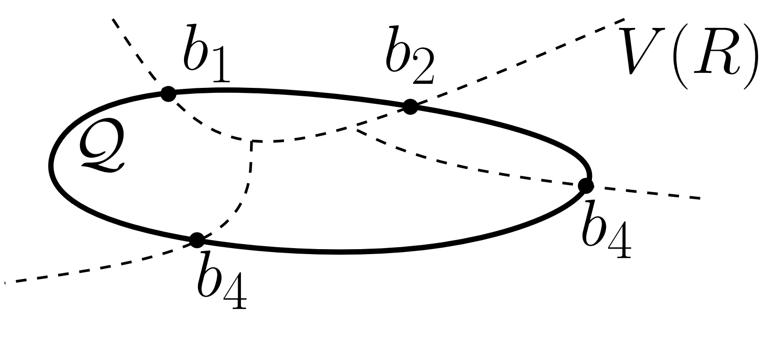
$e - \mathcal{Q}$

Proof

- Assume first $e \cap \mathcal{Q} = V(R) \cap \mathcal{Q}$
  - Since $V(R) \cap \mathcal{Q}$ is connected, $e \cap \mathcal{Q}$ is connected
- Assume next t $e \cap \mathcal{Q} \neq V(R) \cap \mathcal{Q}$
  - At least one endpoint of e is contained in $\mathcal{Q}$
  - For every point $x \in e \cap \mathcal{Q}$, one of the subpaths of e connecting x to an endpoint of e must be contained in $\mathcal{Q}$
  - $e \cap \mathcal{Q}$ or $e - \mathcal{Q}$ is a single component

Rough Idea

- Let L be $\{e \in V(R) \mid (e, s) \in G(R)\}$
- For every edge $e \in L$, let e' be $e - \mathcal{Q} = e - \text{VR}(s, R \cup \{s\})$. If e is an edge between $\text{VR}(p, R)$ and $\text{VR}(q, R)$, $e' = e - D(s, p) = e - D(s, q)$
- Let B be $\{x \in x$ is an endpoint of e' but is not an endpoint of $e\} = V(R) \cap \text{bd } \mathcal{Q}$
- bd Q is a cyclic ordering on the points in B



Step 1: Compute e' for each edge $e \in L$

Step 2: Compute B and cyclic ordering on B induced by bd $\mathcal{Q}$

Step 3: Let $x_1, \ldots, x_k$ be the set B in its cyclic ordering ($x_{k+1} = x_1$), and let r_i such that $(x_i, x_{i+1}) \in \mathrm{VR}(r_i, r)$

- For $1 \leq i \leq k$, add the part of $J(r_i, s)$ with endpoints x_i and x_{i+1}

Lemma 6

$V(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in time $O(\deg_{G(R)}(s)+1)$

Lemma 7

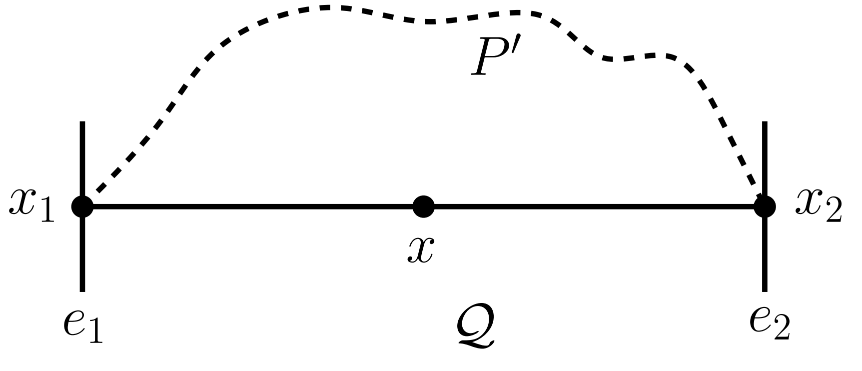
$G(R \cup \{s\})$ can be constructed from $V(R)$ and $G(R)$ in $O(\Sigma_{(e,s)\in G(R)} \deg_{G(R)}(e))$ time

1. Edges of $V(R \cup \{S\})$ which were alreay edges of $V(R)$ don't changes
2. Edges of $V(R \cup \{S\})$ which are parts of edges in L
   - consider each edge $e \in L$
   - If $e \subseteq \mathcal{Q}$, e has to be deleted from conflict graph.
   - If $e \nsubseteq \mathcal{Q}$, $e - \mathcal{Q}$ consists at most two subsegment.
   - let e' be one of the subsegments and let t be a site in $S \setminus R \cup \{s\}$.
   - $e' \cap \mathrm{VR}(t, R \cup \{s, t\}) = e' \cap_{r \in R} D(t, r) \cap D(t, s) = e' \cap \mathrm{VR}(t, R \cup \{t\}) \cap D(t, s) \subseteq e \cap \mathrm{VR}(t, R \cup \{t\})$
   - Any site t in conflict with e' must be in conflict with e
   - Takes time $O(\Sigma_{e\in L} \deg_{G(R)}(e)) = O(\Sigma_{(e,s)\in G(R)} \deg_{G(R)}(e))$
3. Edges of $\mathrm{VR}(s, R \cup \{s\})$ which are complete new
   - Let e_{12} connect x_1 and x_2 in B
   - Let e_{12} belong to $\mathrm{VR}(p, R)$ such that e_{12} belongs to $J(p, s)$
   - Let $x_1 \in e_1$ of $\mathrm{VR}(p, R)$ and $x_2 \in e_2$ of $\mathrm{VR}(p, R)$
   - Let P be the part of bd $\mathrm{VR}(p, R)$ which connects x_1 and x_2 and is contained in cl $\mathcal{Q}$.
   - Lemma 8 will prove that If $t \in S \setminus R \cup \{s\}$ is in conflict with e_{12}, t must be in conflict with either e_1, e_2 or one of the edges of P
   - Each edge in L is involved at most twice, takes time $O(\Sigma_{(e,s)\in G(R)} \deg_{G(R)}(e))$

Lemma 7
Let $t \in S \setminus (R \cup \{s\})$ and let t conflict with e_{12} in $V(R \cup \{s\})$ (as defined in Lemma 7). t conflicts with e_1, e_2, or one of the edges of P.

Proof:

- By the definition of conflict, a point $x \in e_{12}$ exists such that $x \in \mathrm{VR}(t, R \cup \{s,t\} \subseteq \mathrm{VR}(t, R \cup \{t\})$
- Assume the contrary that t does not conflict with e_1, e_2, or one edge of P.
- For any sufficiently small neighborhood of $U(x_1)$ of x_1, $\mathrm{VR}(t, R \cup \{s,t\}) \cap U(x_1) \subseteq \mathrm{VR}(t, R \cup \{t\}) \cap U(x_1) = \emptyset$, and it is also tru for x_2.
- Let p be a site in R such that $e_{12} \subseteq \mathrm{cl}\ \mathrm{VR}(p, R \cup \{s\})$, implying that $x_1, x_2 \in \mathrm{cl}\ \mathrm{VR}(p, R \cup \{s\})$
- There is a path P' from x_1 to x_2 completely inside $\mathrm{VR}(p, R\{s,t\}) \subseteq \mathrm{VR}(p, R \cup \{t\})$.
- The cycle $x_1 \circ P \circ x_2 \circ P'$ contains $\mathrm{VR}(t, R \cup \{t\})$ and is contained in $\mathrm{VR}(p, R \cup \{t\})$.
- contradict $\mathrm{VR}(p, R \cup \{t\})$ is simply connected



Theorem 1
Let $s \in S \setminus R$. $G(R \cup \{s\})$ and $V(R \cup \{s\})$ can be constructed from $G(R)$ and $V(R)$ in time $O(\Sigma_{(e,s) \in G(R)} \deg_{G(R)}(e))$

Theorem 2
$V(S)$ can be computed in $O(n \log n)$ expected time

- $\Sigma_{3 \le i \le n} O(\Sigma_{(e,s_i) \in G(R_{i-1})} \deg_{G(R_{i-1})}(e))$
- Let e be a Voronoi edge of $V(R_i)$ and let s be a site in $S \setminus R_i$ which conflicts e.
- The conflict relation (e, s) will be counted only once since the counting only occured when e is removed
  - Let s_j be the earliest site in the sequence which conflicts with e. Then (e, s) will be counted in $\deg_{G(R_{j-1})}(e)$
- Time proportional to the number of conflict relations between Voronoi edges in $\cup_{2 \le i \le n} V(R_i)$ and sites in S
- The expected size of conflict history is $-C_n + \Sigma_{2 \le i \le n}(n - j + 1)p_j$
  - Kenneth L. Clarkson and Kurt Mehlhorn and Raimund Seidel, "Four Results on Randomized Incremental Constructions," Computational Geometry, vol. 3, no. 4, pp. 185–pp. 212.
  - C_n is the expected size of $\cup_{2 \le i \le n} V(R_i)$
  - p_j is the expected number of Voronoi edges defined by the same two sites in $V(R_j)$
- Since $C_n = O(n)$ and $p_j = O(1/j)$, the expected run time is $O(n \log n)$