Exercise 1: (4 Points)
Recall the Greedy-by-Value and Greedy-by-Sqrt-Value-Density algorithms for single-minded CAs of lecture 12. Let us analyse another greedy algorithm that looks as follows.

Greedy-by-Value-Density

- Re-order the bids such that \( \frac{b_1^*}{|S_1^*|} \geq \frac{b_2^*}{|S_2^*|} \geq \cdots \geq \frac{b_n^*}{|S_n^*|} \).
- Initialize the set of winning bidders to \( W = \emptyset \).
- For \( i = 1 \) to \( n \) do: If \( S_i^* \cap \bigcup_{j \in W} S_j^* = \emptyset \), then \( W = W \cup \{i\} \).

Let \( d = \max_{i \in N} |S_i^*| \). Show that the given algorithm yields a \( d \)-approximation.

Exercise 2: (3 Points)
Recall the auction of \( k \) identical items from Exercise Set 6. Each bidder can acquire at most one of the items. If bidder \( i \) gets one of the items, she has a value of \( v_i \). Otherwise, that is, if she does not get an item, she has a value of 0. Make use of the VCG-results from the lecture in order to design a truthful mechanism for this auction. For this purpose, explicitly state the function \( f \) and calculate the payment rule \( p \).

Exercise 3: (3 Points)
Consider a single-parameter problem and let \( f \) be the function that maximizes \( \sum_i b_i x_i \) among all \( x \in X \) (declared welfare). Show that \( f \) is monotone.

Exercise 4: (4 Points)
As seen in the lecture, let \( f: V \to X \) be a function that maximizes declared welfare, i.e., \( f(b) \in \arg \max_{x \in X} \sum_i b_i(x) \) for all \( b \in V \). For each \( i \), let \( h_i \) be an arbitrary function \( b_{-i} \mapsto h_i(b_{-i}) \) which does not depend on \( b_i \). We define a mechanism \( M = (f, p) \) by setting

\[
p_i(b) = h_i(b_{-i}) - \sum_{j \neq i} b_j(f(b)) .
\]

Prove that \( M \) is a truthful mechanism.
Exercise 5: \hspace{1cm} (4+2 Points)

Consider the following *Procurement Auction*. It’s being attempted to buy a certain item. There are \(n\) vendors who are able to manufacture the wanted item. Vendor \(i\) incurs a cost of \(c_i\) for crafting the item. Now, the vendors are asked to state their costs for crafting the item and a vendor with lowest cost shall be chosen. The latter potentially gets a payment for it. The stated problem can be formalized by the general model of the lecture: Each vendor \(i\) is interpreted as a bidder who has negative valuation \(v_i\), if he/she is chosen to craft the item, that is, \(v_i(x) = -c_i\), if \(i\) is chosen in \(x\).

(a) The results of the lecture concerning VCG are applicable in this situation. Make use of them in order to state a truthful mechanism. Note that this mechanism won’t be *individually rational*.

(b) Make use of the results from Exercise 4 in order to modify the mechanism to be individually rational.