Algorithms and Uncertainty
Winter Semester 2018/19
Exercise Set 7

Exercise 1: (4 Points)
Prove Theorem 15.1: Show that the expected regret of the basic explore-exploit algorithm is at most \( O(nT^{\frac{2}{3}}\sqrt{\ln(nT)}) \).

**Hint:** Use Hoeffding's inequality and the union bound to upper-bound the probability that there is an \( i \) for which \( \hat{\mu}_i > \mu_i + \epsilon \) for a suitably chosen \( \epsilon \).

Exercise 2: (4 Points)
We consider a generalization of the algorithm Weighted Majority for classifiers with \( k \) different classes. (The case covered in the lecture, binary classification, is \( k = 2 \).) In each step, the algorithm chooses a class, which is recommended by the largest number of classifiers (so the class has a plurality).
Show that this algorithm makes at most \((2 + \eta)\min m_i + 2 \ln n/\eta \) errors, where \( m_i \) is the number of errors of classifier \( i \).

Exercise 3: (4 Points)
Consider the modified update rule for Multiplicative Weights that sets \( w_i^{(t+1)} = w_i^{(t)} \cdot (1 - \ell_i^{(t)} \eta) \).
Show that Theorem 16.3 still holds.

Exercise 4: (4 Points)
Show that every no-regret algorithm in the experts setting has to be randomized. Consider the case \( n = 2 \) and for every deterministic algorithm construct a sequence such that \( L^{(T)}_{\text{Alg}} = T \) and \( \min_i L_i^{(T)} \leq T/2 \).

Exercise 5: (4 Points)
Use Yao's principle to show that for every (randomized) algorithm there is a sequence \( \ell^{(1)}, \ldots, \ell^{(T)} \) such that \( L^{(T)}_{\text{Alg}} \geq (1 - \frac{1}{n}) T \) but \( \sum_{t=1}^{T} \min_i \ell_i^{(t)} = 0 \). This shows that the order of sum and maximum in the regret definition are important.

Bonus:
Use the one-sided version of Hoeffding’s inequality to show a regret bound for UCB1 of \( \sum_{i \neq i^*} \frac{2 \ln T}{\Delta_i} + 2 \Delta_i \). The one-sided version of Hoeffding’s inequality is as follows: Let \( Z_1, \ldots, Z_N \) be independent random variables such that \( a_i \leq Z_i \leq b_i \) with probability 1. Let \( Z = \frac{1}{N} \sum_{i=1}^{N} Z_i \) be their average. Then for all \( \gamma \geq 0 \)

\[
\Pr \left[ Z - \mathbb{E}[Z] \geq \gamma \right] \leq \exp \left( -\frac{2N^2 \gamma^2}{\sum_{i=1}^{N} (b_i - a_i)^2} \right).
\]