1 Convex hulls of random point sets

We consider a set $P$ of $n$ points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in $[0, 1]^2$, where the coordinates are chosen uniformly and independently at random from $[0, 1]$.

(a) Consider a random assignment to $x_1, \ldots, x_n, y_1, \ldots, y_n$ as defined above. Assume that all $x_i$ and $y_i$ are different. Let $\sigma : [n] \to [n]$ be the permutation such that $x_{\sigma(i)} < x_{\sigma(j)}$ for all $i < j$. Derive an upper bound on the probability that the $r$-th point from the left, $(x_{\sigma(r)}, y_{\sigma(r)})$, lies on the convex hull.

(b) Show that the expected number of vertices of the convex hull of $P$ is in $O(\log n)$.

Hint: You may use the fact that all sorted orders of the random variables $x_i$ and $y_i$ are equally likely. More precisely, let $\tau : [n] \to [n]$ be the permutation such that $y_{\sigma(\tau(i))} < y_{\sigma(\tau(j))}$ for all $i < j$. In words, $\tau^{-1}$ is the permutation that would turn the list of points, sorted by $x$-coordinates, into a list sorted by $y$-coordinates. You may use the fact that all permutations $\tau$ are equally likely, for any $\sigma$.

2 Faster computation of the convex hull

Consider a set $P \subset \mathbb{R}^3$ of $n$ points. Assume that if we pick a random sample $Y \subset P$ of size $r$ then the expected number of points in $Y$ on the boundary of $\text{conv}(Y)$ is $O(r^\alpha)$, for some constant $\alpha < 1$. Prove that under this condition, the expected running time of the convex hull algorithm given in the lecture is in $O(n)$.

3 Delaunay triangulations and Voronoi diagrams

a) Define a graph on a set $P \subset \mathbb{R}^2$ as follows: Two points $p$ and $q$ are connected by an edge if and only if there exists a disk with both $p$ and $q$ on the boundary and with no point of $P$ in its interior. Prove that this graph is the Delaunay triangulation of $P$.

b) Given a Delaunay triangulation of $P \subset \mathbb{R}^2$ as a doubly connected edge list (DCEL), compute the graph of the Voronoi Diagram of $P$. Use the definition of the Voronoi diagram that includes an additional vertex at infinity that is incident to all unbounded Voronoi edges. Use the following definition of a DCEL:

- The vertex record of a vertex $v$ stores the coordinates of $v$ in $\text{Coordinates}(v)$. It also stores a pointer $\text{IncidentEdge}(v)$ to an arbitrary half-edge that has $v$ as its origin.
- The face record of a face $f$ stores a pointer $IncidentEdge(f)$ to some half-edge on its boundary (this also holds for the outer face).

- The half-edge record of a half-edge $\vec{e}$ stores a pointer $Origin(\vec{e})$ to its origin, a pointer $Twin(\vec{e})$ to its twin half-edge, and a pointer $IncidentFace(\vec{e})$ to the face that it bounds. The origin is chosen such that $IncidentFace(\vec{e})$ lies to the left of $\vec{e}$ when it is traversed from origin to destination. The half-edge record also stores pointers $Next(\vec{e})$ and $Prev(\vec{e})$ to the next and previous edge on the boundary of $IncidentFace(\vec{e})$.

c) Show that the vertical projection of the edges of the polytope in $\mathbb{R}^3$ constructed via the lifting map as in the lecture is the Voronoi diagram of $P$. 