1 Maximum level in an arrangement

Given a set $L$ of $n$ lines in the plane, the level of a point $x \in \mathbb{R}^2$ is the number of lines lying strictly above $x$. Give an $O(n \log n)$ time algorithm to compute the maximum level of any vertex in the arrangement $A(L)$, where $L$ is not necessarily in general position.

2 Vapnik-Chervonenkis dimension

1. Consider the range space $S = (X, \mathcal{R})$. The complement of $S$ is defined as the space $\overline{S} = (X, \overline{\mathcal{R}})$, where $\overline{\mathcal{R}} = \{X \setminus r \mid r \in \mathcal{R}\}$. Show how the VC dimension of $\overline{S}$ is related to the VC dimension of $S$.

2. Let $S_1 = (X, \mathcal{R}_1)$, $S_2 = (X, \mathcal{R}_2)$ be two range spaces with VC dimension $\delta_1$ and $\delta_2$ respectively. Let $\hat{\mathcal{R}} = \{r_1 \cap r_2 \mid r_1 \in \mathcal{R}_1, r_2 \in \mathcal{R}_2\}$. Show that the VC dimension of $(X, \hat{\mathcal{R}})$ is $O((\delta_1 + \delta_2) \log(\delta_1 + \delta_2))$.

3. Consider a range space $S = (X, \mathcal{R})$, where $X$ is a finite subset of $\mathbb{R}^2$ and $\mathcal{R} = \{\Delta \cap X \mid \Delta \text{ is a triangle in } \mathbb{R}^2\}$. Show an upper bound on the VC dimension of $S$.

4. Consider a range space $S = (X, \mathcal{R})$, where $X$ is a finite subset of $\mathbb{R}^2$ and $\mathcal{R} = \{\Pi_k \cap X \mid \Pi_k \text{ is a convex } k \text{-gon in } \mathbb{R}^2\}$, for a given integer $k$. Show an upper bound on the VC dimension of $S$. 