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Randomized Algorithms and
Probabilistic Analysis
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Problem Set 8

Problem 1

Show that the recurrence

$$h_0 = 1 + h_1, \quad h_n = 0, \quad \forall j \in \{1, \ldots, n-1\} : h_j \le 1 + \frac{2}{3}h_{j-1} + \frac{1}{3}h_{j+1}$$

implies that $h_j \le 2^{n+2} - 2^{j+2} - 3(n-j)$ for all $j \in \{0, \ldots, n\}$. *Hint:* First show by induction that for all $j \in \{0, \ldots, n-1\}$, $h_j \le h_{j+1} + 2^{j+2} - 3$ holds.

(This task completes the proof of Lemma 4.7).

Problem 2

The Nemhauser-Ullmann algorithm is not the algorithm we usually learn for the knapsack problem in bachelor's courses on algorithms. Indeed, the knapsack problem can be solved by an algorithm with running time $\mathcal{O}(nW)$, where we assume that the input consists of n items with deterministic integral profits $p_1, \ldots, p_n \ge 0$, deterministic integral weights $w_1, \ldots, w_n \ge 0$ and a capacity $W \le \sum_{i=1} w_i$. Briefly describe/explain an algorithm that achieves this. What does its existence mean for the complexity of the knapsack problem?

Problem 3

We claimed that the Nemhauser-Ullmann can be implemented to run in time $\Theta(\sum_{i=0}^{n-1} |\mathcal{P}_i|)$ (Theorem 6.4) . Finish the proof of Theorem 6.4 and show how the solution can be reconstructed from the sets $val(\mathcal{P}_i)$.

We highly recommend implementing the Nemhauser-Ullmann algorithm with running time $\Theta(\sum_{i=0}^{n-1} |\mathcal{P}_i|)$ as an exercise.

Problem 4

Let's familiarize ourselves a bit with continuous probability spaces, which are defined and discussed in Section 5 of the lecture notes.

Assume that we have a continuous roulette wheel. We spin it and then it halts at a position P between $[0, 1]$. The position is chosen uniformly at random, i.e., P is uniformly distributed on $[0, 1]$.

1. What is the probability to get the number 0.2514?
2. What is $\mathbf{E}[P]$? More generally, let Q be uniformly distributed over $[a, b]$ and compute $\mathbf{E}[Q]$.

Now we spin the wheel twice. Let Y_1, Y_2 be the outcomes. We assume that Y_1 and Y_2 are independent, both uniformly distributed on $[0, 1]$. Let $Y_{\min} := \min\{Y_1, Y_2\}$ be the smaller number of the two runs.

3. What is the probability that $Y_{\min} \geq y$ for an $y \in [0, 1]$?
4. What is the probability that $Y_{\min} \geq y$ for an $y \in (1, \infty]$?
5. Compute $\mathbf{E}[Y_{\min}]$.
6. Compute $\mathbf{E}[Y_{\max}]$, where $Y_{\max} = \max\{Y_1, Y_2\}$.