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9th April 2018
Due date: 16th April 2018, 10:15

Algorithmic Game Theory and the Internet

Summer Term 2018

Exercise Set 1

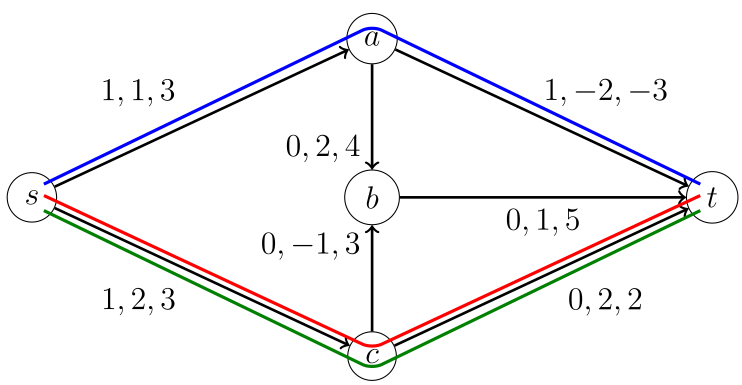
Please hand in your solutions at the beginning of the lecture on 16th of April. You may work in groups of up to three students. Please write down your name(s) on the solutions and tag them as *am* or *pm*, if you are going to participate in the exercises in the morning or afternoon on Thursdays, respectively. The solutions can be written down in German or English.

Exercise 1: (2+4 Points)

Consider the following symmetrical network congestion game with players blue, red and green and their corresponding beginning strategies.

a) Formalize the network congestion game depicted below. For this purpose, specify the tuple $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i\in\mathcal{N}}, (d_r)_{r\in\mathcal{R}})$ like in the Example 1.2 of the first lecture. It suffices to state the delay function of a single resource/edge.

b) Find a pure Nash equilibrium by stating a sequence of best response improvement steps.



Exercise 2: (2 Points)

Consider congestion games with a constant number of players. Show that the length of every sequence of improvement steps is bounded polynomially in the number of player strategies.

Exercise 3: (6 Points)

Give an example for a symmetrical network congestion game (strategies are s-t paths in a directed graph with the same s and t for all players) with monotonically increasing delay functions d_r such that there exist at least two pure Nash equilibria with different *social costs*. We define the social cost to be the sum of all players' costs $\sum_{i\in\mathcal{N}} c_i(S)$.

Exercise 4 on the next page.

Exercise 4: (5+1 Points)

In a *weighted* congestion game, every player $i \in \mathcal{N}$ has an individual weight $w_i > 0$. The delay of a resource r now depends on the sum of the weights – instead of the number of players – of those players who are using r. For this purpose, we could redefine $n_r(S)$ to be $n_r(S) = \sum_{i:r \in S_i} w_i$ and consider delay functions $d_r\colon \mathbb{R}_{\geq 0} \to \mathbb{R}$. Like in the unweighted case the cost of player i is defined as $c_i(S) = \sum_{r \in S_i} d_r(n_r(S))$.

(a) Prove that weighted congestion games do not fulfil the *Finite-Improvement Property*, even having only two players, three resources and two strategies for each player.

Hint: Consider $\mathcal{N} = \{1, 2\}$, $w_1 = 1$, $w_2 = 2$, $R = \{a, b, c\}$, $\Sigma_1 = \{\{a\}, \{b, c\}\}$, $\Sigma_2 = \{\{b\}, \{a, c\}\}$. Choose delay functions such that

$$(\{a\}, \{b\}) \to (\{a\}, \{a, c\}) \to (\{b, c\}, \{a, c\}) \to (\{b, c\}, \{b\}) \to (\{a\}, \{b\})$$

is a sequence of best response improvement steps.

(b) Show with the aid of part (a) that there does not have to exist a pure Nash equilibrium.