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Due date: In the lecture on 4th June 2018, 10:15

Algorithmic Game Theory and the Internet

Summer Term 2018

Exercise Set 7

Exercise 1: (5 Points)

Recall the *Greedy-by-Value* and *Greedy-by-Sqrt-Value-Density* algorithms for single-minded CAs of lecture 13. Let us analyse another greedy algorithm that looks as follows.

Greedy-by-Value-Density

- Re-order the bids such that $\frac{b_1^*}{|S_1^*|} \geq \frac{b_2^*}{|S_2^*|} \geq \cdots \geq \frac{b_n^*}{|S_n^*|}$.
- Initialize the set of winning bidders to $W = \emptyset$.
- For $i = 1$ to n do: If $S_i^* \cap \bigcup_{j \in W} S_j^* = \emptyset$, then $W = W \cup \{i\}$.

Let $d = \max_{i \in \mathcal{N}} |S_i^*|$. Show that the given algorithm yields a d-approximation.

Exercise 2: (5 Points)

Consider a *Knapsack Auction* which is defined the following way. Each bidder i has a publicly known weight w_i and a private value v_i. A feasible outcome is any set S of bidders such that $\sum_{i \in S} w_i \leq W$ holds for a fixed bound W. Furthermore, we assume that $0 \leq w_i \leq W$ for all bidder i.

The following algorithm yields a 2-approximation:

- Sort and renumber the bidders such that $\frac{b_1}{w_1} \geq \frac{b_2}{w_2} \geq \ldots \frac{b_n}{w_n}$. Let k be the largest integer such that $\sum_{i=1}^{k} w_i \leq W$ and set $S_1 = \{1, \ldots, k\}$.
- Let i^* be the bidder with the maximum bid b_i among all bidders and set $S_2 = \{i^*\}$.
- Return the better solution of S_1 and S_2.

Show that the given algorithm is monotone and state a truthful mechanism with the aid of Myerson's Lemma.

Exercise 3 on the next page.

Exercise 3: (3+3+4 Points)
In this exercise we want to prove an alternative characterization of truthful mechanisms which is stated in the following theorem. Moreover, this restated characterization holds for arbitrary mechanisms and not only for single-parameter mechanisms.

Theorem. A mechanism $M = (f, p)$ is truthful if and only if the following two conditions are met.

(i) For every pair b_i, b_i': If $f(b_i, b_{-i}) = f(b_i', b_{-i})$, then we also have $p_i(b_i, b_{-i}) = p_i(b_i', b_{-i})$. In other words: For all b_{-i}, for all $a \in X$ there exist prices $p_a \in \mathbb{R}$ such that for all b_i with $f(b_i, b_{-i}) = a$ we have $p_i(b_i, b_{-i}) = p_a$.

(ii) The mechanism optimizes for each player. Formally: For every pair b_i, b_{-i} the following holds
$$f(b_i, b_{-i}) \in \arg\max_{a \in A} (b_i(a) - p_a),$$
where the set of allocations A is equal to the image of $f(\cdot, b_{-i})$.

For this purpose, prove the following claims:

(a) If condition (i) is violated, then the mechanism M cannot be truthful.

(b) If condition (ii) is violated, then the mechanism M cannot be truthful.

(c) If conditions (i) and (ii) are met, then M is a truthful mechanism.

Hint: You should not use Myerson's Lemma in this task (since it is not helpful). Instead, it is sufficient to make use of the truthfulness inequality with reasonably chosen deviations. See also how we derived *payment difference sandwich* in the proof of Theorem 12.2.