

**Algorithmic Game Theory and the Internet**  
Summer Term 2018  
Exercise Set 10

**Exercise 1:** (4+4 Points)

Recall the valuation functions of single-minded bidders from Definition 13.2. Let the maximum bundle size be defined by  $d = \max_{i \in \mathcal{N}} |S_i^*|$ .

- (a) Show that in the case of single-minded bidders with maximum bundle size  $d$ , item bidding with first price payments is  $(\frac{1}{2}, 2d)$ -smooth.

**Hint:** In order to define deviation bids  $b_{i,j}^*$ , consider a welfare-maximization allocation on  $v$ . If bidder  $i$  does not get his bundle in the optimal allocation, then define  $b_{i,j}^* = 0$  for all items  $j \in M$ . Otherwise, define  $b_{i,j}^* = \frac{v_i}{2d}$  for all  $j \in S_i^*$  and  $b_{i,j}^* = 0$  if  $j \notin S_i^*$ . That is, each winner in the optimal allocation equally divides the value for his bundle among all items of the bundle and bids half of it.

- (b) Now, we define prices for items as in Lecture 20 by setting

$$p_j^v = \begin{cases} \frac{1}{2d} v_i(S_i^*) & \text{if buyer } i \text{ gets item } j \text{ in optimal solution on } v \\ 0 & \text{if item } j \text{ is unassigned in optimal solution on } v \end{cases}$$

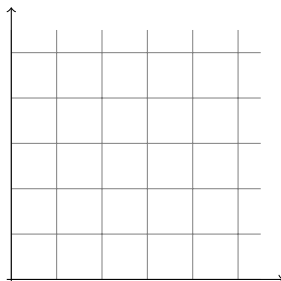
Show that using these prices in the full-information setting gives a  $\frac{1}{2d}$ -approximation of the optimal social welfare. (Like in Step 1 of Lecture 20)

**Exercise 2:** (2+3 Points)

Consider three unit-demand buyers and two items with

$$v_{1,1} = 5, v_{1,2} = 3, v_{2,1} = 3, v_{2,2} = 4, v_{3,1} = 2, v_{3,2} = 2 .$$

- (a) Determine the Walrasian price vector that as determined by the VCG mechanism.
- (b) Now find *all* Walrasian price vectors  $q$ . (We know that the solution to (a) is component-wise smaller than any other such vector.) Draw these vectors in a coordinate system with axes  $q_1$  and  $q_2$ .



Exercises 3 and 4 on the next page.

**Exercise 3:**

(4 Points)

Consider  $m$  items and  $n$  unit-demand bidders. We define a generalization of Walrasian equilibria: Let  $S$  be a matching of items to bidders and  $q \in \mathbb{R}_{\geq 0}^m$  be a price vector. We call the pair  $(q, S)$  an  $\epsilon$ -approximate Walrasian equilibrium if unallocated items have price 0, every bidder  $i$  has non-negative utility  $v_{i,S(i)} - q_{S(i)} \geq 0$ , and every bidder receives an item within  $\epsilon$  of its favorite, i.e.,  $v_{i,S(i)} - q_{S(i)} \geq v_{i,j} - q_j - \epsilon$  for every item  $j$ .

Prove an approximate version of the First Welfare Theorem: If  $(q, S)$  is an  $\epsilon$ -approximate Walrasian equilibrium, then the social welfare of an optimal matching  $S^*$  cannot surpass the one of  $S$  by more than  $\min\{m, n\} \cdot \epsilon$ .

**Exercise 4:**

(3 Points)

Have a look at the single-minded combinatorial auction with three bidders and items  $a, b, c$  which is depicted below. State all values of  $x \in \mathbb{R}_{\geq 0}$  such that there exists a Walrasian equilibrium and prove your claim.

