MA-INF 1203 Discrete and Computational Geometry

Wintersemester 2019/20

Assignment 4

Deadline: **5** November before noon (To be discussed: **5/6**. November 2019)

1 Duality

a) Let $X \subset \mathbb{R}^d$ be a closed convex set containing the origin. Prove that $X = (X^*)^*$.

b) For a finite set $X \subset \mathbb{R}^d$, prove that $((X)^*)^* = \text{conv}(X \cup \{0\})$.

2 Lifting

a) Let $D \subset \mathbb{R}^3$ be any disk in the hyperplane given by the equation $x_3 = 0$. A disk which belongs to the hyperplane $x_3 = 0$, with center $(a_1, a_2, 0) \in \mathbb{R}^3$, and radius $r > 0$ is equal to the set $\{(x_1, x_2, 0) \in \mathbb{R}^3 \mid (x_1 - a_1)^2 + (x_2 - a_2)^2 \leq r^2\}$. Show that there exists a half-space h such that D is the vertical projection of the set $h \cap U$ onto $x_3 = 0$ where $U = \{x \in \mathbb{R}^3 \mid x_3 = x_1^2 + x_2^2\}$ is the unit paraboloid.

b) Consider n arbitrary circular disks $D_1, \ldots, D_n$ in the plane. Show that there exist only $O(n)$ intersections of their boundaries that lie inside no other D_i (this means that the boundary of the union of the D_i consists of $O(n)$ circular arcs).

3 Voronoi diagrams

Let $d = 2k + 1$ be odd, let $e_1, \ldots, e_d$ be vectors of the standard orthonormal basis in $\mathbb{R}^d$, and let e_0 stand for the zero vector. For $i = 0, 1, \ldots, k$ and $j = 1, 2, \ldots, n$, let $p_{i,j} = e_{2i} + \frac{j}{n} e_{2i+1}$.

a) Prove that for every choice of $j_0, j_1, \ldots, j_k \in \{1, 2, \ldots, n\}$, there is a point q in $\mathbb{R}^d$ with

$$q = \left(\frac{j_0}{n}, x_1, \frac{j_1}{n}, x_2, \frac{j_2}{n}, \ldots, x_k, \frac{j_k}{n} \right)$$

for some real values $x_1, \ldots, x_k$ such that the points which are nearest to q among the $p_{i,j}$ are exactly $p_{0,j_0}, p_{1,j_1}, \ldots, p_{k,j_k}$.

b) Conclude that the total number of faces of all dimensions in the Voronoi diagram of the $p_{i,j}$ is $\Omega(n^{k+1}) = \Omega(n^{\lceil d/2 \rceil})$.