Simple Mechanisms for Combinatorial Auctions

Thomas Kesselheim **Preview** Last Update: May 29, 2019

Today, we continue our discussion of simple, non-truthful mechanisms. We consider combinatorial auctions, so there are m items M, which can each be allocated at most once. Bidders have valuation functions $v_i\colon 2^M \to \mathbb{R}_{\geq 0}$.

A *mechanism* $\mathcal{M} = (f, p)$ defines a set of bids B_i for each player $i \in N$ and consists of an *outcome rule* $f\colon B \to X$, where $B = B_1 \times B_2 \times \cdots \times B_n$, and a *payment rule* $p\colon B \to \mathbb{R}^n_{\geq 0}$.

Last time, we introduced the definition of a smooth mechanism.

Definition 16.1 (Smooth Mechanism, simplified version)**.** *Let $\lambda, \mu \geq 0$. A mechanism $\mathcal{M}$ is (λ, μ)-smooth if for any valuation profile $v \in V$ for each player $i \in \mathcal{N}$ there exists a bid b_i^* such that for any profile of bids $b \in B$ we have*

$$\sum_{i \in \mathcal{N}} u_i(b_i^*, b_{-i}) \geq \lambda \cdot OPT(v) - \mu \sum_{i \in \mathcal{N}} p_i(b) \quad .$$

It is easy to see that (λ, μ)-smoothness implies that the Price of Anarchy for pure Nash equilibria is at most $\frac{\max\{\mu, 1\}}{\lambda}$. This proof also generalizes to (coarse) correlated equilibria. In a more complex argument, we were also able to show that the bound also holds for Bayes-Nash equilibria. Given these results, it is enough to show smoothness of mechanisms to bound the Price of Anarchy for all equilibrium concepts that we introduced so far. Interestingly, all results that we cover today were discovered before the smoothness result was discovered, but the basic arguments were already present in the original publications.

1 Item Bidding

We first consider a truly simple, indirect mechanism. Instead of reporting complex functions $2^M \to \mathbb{R}_{\geq 0}$, the bidders now simply report a single bid $b_{i,j}$ for each item j. Each item is sold in a separate first-price or second price-auction. That is, item j is assigned to the bidder i with the highest bid $b_{i,j}$. He has to pay $b_{i,j}$.

A bidder can potentially win multiple items, even if he only wants one. Recall *unit-demand valuations*: These are functions v_i such that there are $v_{i,j} \in \mathbb{R}_{\geq 0}$ such that $v_i(S) = \max_{j \in S} v_{i,j}$. If, for example, $v_{i,1} = \ldots = v_{i,m} = 1$, then bidder i has a value of 1 as long as he receives an item, no matter which. There is no way to express this in a bid. Therefore, this is not a direct mechanism and it cannot be truthful. However, its Price of Anarchy is bounded by 2.

Theorem 16.2. *For unit-demand valuations, item bidding with first-price payments is $(\frac{1}{2}, 1)$-smooth.*

Proof. We have to devise the deviation bids b_i^* for all bidders. These bids may depend on the valuations v but not on the bids. Consider the welfare-maximization allocation on v. Let j_i be the item that is assigned to bidder i in this allocation. If i does not get any item, set j_i to $\perp$.

We now set $b_{i,j}^* = \frac{v_{i,j}}{2}$ if $j = j_i$ and 0 otherwise. That is, in the deviation bid, each bidder bids half his value on the item that he is supposed to get.

Given any bid profile b, bidder i's utility after deviating is $\frac{v_{i,j_i}}{2}$ unless another bidder bids at least $\frac{v_{i,j_i}}{2}$ for item j_i in b. Therefore

$$u_i((b_i^*, b_{-i}), v_i) \geq \frac{v_{i,j_i}}{2} - \max_{i' \neq i} b_{i',j_i} \geq \frac{v_{i,j_i}}{2} - \max_{i'} b_{i',j_i} \quad .$$

If we take the sum over all bidders i, then

$$\sum_{i\in N} u_i((b_i^*, b_{-i}), v_i) \geq \sum_{i\in N} \frac{v_{i,j_i}}{2} - \sum_{i\in N} \max_{i'} b_{i',j_i} \ .$$

Observe that $\sum_{i\in N} v_{i,j_i} = OPT(v)$ because of the way we defined j_i. Furthermore, we have $\sum_{i\in N} \max_{i'} b_{i',j_i} \leq \sum_{j\in M} \max_{i'} b_{i',j} = \sum_{i\in N} p_i(b)$ because every item is counted at most once: For each item j there is at most one i such that $j = j_i$. That is,

$$\sum_{i\in N} u_i((b_i^*, b_{-i}), v_i) \geq \frac{1}{2} OPT(v) - \sum_{i\in N} p_i(b) \ ,$$

which is exactly $(\frac{1}{2}, 1)$-smoothness. □

So, immediately we get that the Price of Anarchy for pure Nash equilibria is at most 2.

2 A Greedy Mechanism

Instead of selling items individually, one can also apply a smarter allocation algorithm and use a direct mechanism. We will now consider a mechanism based on the Greedy-by-Sqrt-Density algorithm for combinatorial auctions. We introduced it as algorithm for single-minded bidders. That is, each bidder is only interested in a single set of items. Under these circumstance, it can be turned into a truthful mechanism. Beyond this single-parameter domain, it cannot be turned into a truthful mechanism. However, as we will show, it can be turned into a mechanism of reasonable Price of Anarchy.

We assume that bidders report functions $b_i \colon 2^M \to \mathbb{R}_{\geq 0}$. (To ensure polynomial running time, only a polynomial number of bundles should have a positive value.) On the pairs (i, S) we run the greedy allocation rule. Each bidder gets only one such bundle S. If the mechanism wanted to allocate not only S to i but also S', it would have to select the pair $(i, S \cup S')$.

We combine this with a first-price payment rule: If bidder i gets set S, then his payment is exactly his bid on this set $b_i(S)$.

First-Price Greedy Mechanism for Combinatorial Auctions

1. Collect bids b.
2. Sort the player-bundle pairs (i, S) by non-increasing score $\frac{b_i(S)}{\sqrt{|S|}}$.
3. Go through the sorted list and assign S to player i unless
   (a) player i has already been allocated a bundle or
   (b) one or more of the items in S has already been allocated.
4. Charge each player i his bid $b_i(S)$ on the bundle S he is allocated.

Theorem 16.3 (Borodin and Lucier, 2010)**.** *The first-price greedy mechanism for multi-minded CAs is* $(1/2, O(\sqrt{m}))$*-smooth.*

Proof. Let $(X_1^*, \ldots, X_n^*)$ be an allocation that maximizes social welfare. That is, $OPT(v) = \sum_{i\in\mathcal{N}} v_i(X_i^*)$. For each player $i \in \mathcal{N}$ let b_i^* be the single-minded declaration for set X_i^* at value $v_i(X_i^*)/2$. So, by bidding b_i^*, bidder i only tries to win the set that he is allocated in the social optimum.

Consider an arbitrary bid profile b. We know that the algorithm is monotone on single-minded bids. That is, if bidder i reports that he is only interested in set S, then there is a smallest bid with which player i wins bundle S against bids b_{-i}. Call this the *critical bid* $\tau_i(S, b_{-i})$.

In particular, bidding b_i^* against b_{-i}, bidder i may or may not win the set X_i^*. If he wins then $u_i((b_i^*, b_{-i}), v_i) = v_i(X_i^*) - v_i(X_i^*)/2 = v_i(X_i^*)/2$. If he loses, then the critical bid is at least $v_i(X_i^*)/2$. So in either case,

$$u_i(b_i^*, b_{-i}) \geq \frac{1}{2} v_i(X_i^*) - \tau_i(X_i^*, b_{-i}) \enspace .$$

Summing over all players $i \in \mathcal{N}$ we obtain

$$\sum_{i \in \mathcal{N}} u_i(b_i^*, b_{-i}) \geq \sum_{i \in \mathcal{N}} \left(\frac{v_i(X_i^*)}{2} - \tau_i(X_i^*, b_{-i}) \right) = \frac{1}{2} \cdot OPT(v) - \sum_{i \in \mathcal{N}} \tau_i(X_i^*, b_{-i}) \enspace .$$

Below, we will show the following lemma.

Lemma 16.4. *Fix bids $b \in B$. Let $f(b)$ be the allocation chosen by the greedy mechanism for bids b and let X^* be another feasible allocation. Then,*

$$\sum_{i \in N} \tau_i(X_i^*, b_{-i}) \leq O(\sqrt{m}) \sum_{i \in \mathcal{N}} b_i(f_i(b)) \enspace .$$

Once we have this lemma, we get

$$\begin{aligned} \sum_{i \in \mathcal{N}} u_i(b_i^*, b_{-i}) &\geq \frac{1}{2} \cdot OPT(v) - O(\sqrt{m}) \cdot \sum_{i \in \mathcal{N}} b_i(f_i(b)) \\ &= \frac{1}{2} \cdot OPT(v) - O(\sqrt{m}) \cdot \sum_{i \in \mathcal{N}} p_i(b) \enspace , \end{aligned}$$

where the last step uses that the mechanism is a first-price mechanism. □

Note that apart from Lemma 16.4 this proof is actually pretty generic. It looks exactly like the smoothness proof for a first-price auction and uses hardly any property of the mechanism. It still remains to prove Lemma 16.4, which indeed relies on the mechanism using a greedy rule.

Proof of Lemma 16.4. Let $\epsilon > 0$. For all i, let b_i^* be the single-minded declaration for set X_i^* at value $\tau_i(X_i^*, b_{-i}) - \epsilon$. Let b_i' be the point-wise maximum of b_i and b_i^*. A crucial property of the greedy algorithm is that the allocation it chooses on profile b' is the same as on b. The reason is that all introduced new bids are below the respective critical bids. Some pairs (i, S) move towards the front in the sorted list. However, none of them moves beyond the point at which it gets accepted. So, its presence does not have any influence of the algorithm. So, formally, $f(b) = f(b')$. Besides, if $b_i(S) \neq b_i^*(S)$ for a set S, then bidder i does not get set S in $f(b)$ or $f(b')$.

That is,

$$\sum_{i \in \mathcal{N}} b_i(f_i(b)) = \sum_{i \in \mathcal{N}} b_i(f_i(b')) = \sum_{i \in \mathcal{N}} b_i'(f_i(b')) \enspace .$$

Now we use the fact that the algorithm is an $O(\sqrt{m})$-approximation. As X^* is a feasible allocation, we have

$$\sum_{i \in \mathcal{N}} b_i'(f_i(b')) \geq \frac{1}{O(\sqrt{m})} \sum_{i \in \mathcal{N}} b_i'(X_i^*) \enspace .$$

By definition of b_i', we also have

$$\sum_{i \in \mathcal{N}} b_i'(X_i^*) = \sum_{i \in \mathcal{N}} \max\left\{ b_i(X_i^*), \tau_i(X_i^*, b_{-i}) - \epsilon \right\} \geq \sum_{i \in \mathcal{N}} (\tau_i(X_i^*, b_{-i}) - \epsilon) = \sum_{i \in \mathcal{N}} \tau_i(X_i^*, b_{-i}) - n\epsilon \enspace .$$

So, in combination

$$\sum_{i \in \mathcal{N}} b_i(f_i(b)) \geq \frac{1}{O(\sqrt{m})} \sum_{i \in \mathcal{N}} \tau_i(X_i^*, b_{-i}) - n\epsilon \enspace .$$

This holds for all $\epsilon > 0$. The claim follows by taking the limit as $\epsilon \to 0$. □

3 Second-Price Auctions

Our results so far were for generalization of the first-price auction. Maybe it would be more natural to generalize the second-price auction. In the case of item bidding this would mean that each item is sold in a separate single-item auction. For the greedy mechanism, we could charge every player the respective critical bid. These are particularly interesting mechanisms because they are truthful in special cases. So, giving a Price-of-Anarchy analysis would show that they are robust beyond the truthful dominant-strategy equilibrium.

Unfortunately, the techniques that we have learned up to now are not enough to bound the Price of Anarchy even for the second-price auction. This is for a good reason: Without further assumptions, it is unbounded.

Observation 16.5. *Consider a single-item second-price auction with two bidders of values* $v_1 = 1$*,* $v_2 = \epsilon$ *for some small* ϵ*. Now* $b_1 = 0$*,* $b_2 = 1$ *is pure Nash equilibrium. Its social welfare is* ϵ *compared to optimal social welfare* 1.

The reason why we get this bad equilibrium is that *overbidding is only weakly dominated.* So, bidders cannot *increase* their utility by overbidding but this does not mean that it decreases.

This is also true in general item bidding with second-price auctions.

Theorem 16.6. *Consider a pure Nash equilibrium* b *of item bidding with second-price payments and unit-demand bidders. Let* $X_1, \ldots, X_n$ *be the resulting allocation. If for all bidders* i *we have* $\sum_{j \in X_i} b_{i,j} \leq v_i(X_i)$ *(weak no-overbidding), then* $\sum_{i \in \mathcal{N}} v_i(X_i) \geq \frac{1}{2} OPT(v)$.

Proof. We mostly follow the steps in the proof of Theorem 16.2. Again, we devise a deviation bid b_i^* for every bidders. Consider the welfare-maximization allocation on v. Let j_i be the item that is assigned to bidder i in this allocation. If i does not get any item, set j_i to $\perp$.

This time, we set $b_{i,j}^* = v_{i,j}$ if $j = j_i$ and 0 otherwise.

Bidder i's utility in (b_i^*, b_{-i}) is $v_{i,j_i} - \max_{i' \neq i} b_{i',j_i}$ if he wins the item, otherwise it is 0 but in this case $\max_{i' \neq i} b_{i,j_i} \geq v_{i,j_i}$. That is, we always have

$$u_i((b_i^*, b_{-i}), v_i) \geq v_{i,j_i} - \max_{i' \neq i} b_{i',j_i} \geq v_{i,j_i} - \max_{i'} b_{i',j_i} \quad .$$

By the equilibrium property $u_i(b, v_i) \geq u_i((b_i^*, b_{-i}), v_i)$. So, taking the sum over all bidders

$$SW_v(b) \geq \sum_{i \in N} u_i(b, v_i) \geq \sum_{i \in N} v_{i,j_i} - \sum_{i \in N} \max_{i'} b_{i',j_i} \quad .$$

Again, $\sum_{i \in N} v_{i,j_i} = OPT(v)$ and $\sum_{i \in N} \max_{i'} b_{i',j_i} \leq \sum_{j \in M} \max_i b_{i,j}$.

However, now $\sum_{j \in M} \max_i b_{i,j}$ does not necessarily have to do anything with the payments. For every item $\max_i b_{i,j}$ may be a lot higher than what the winner has to pay for it. Here, the weak no-overbidding assumption comes to our rescue. We can write

$$\sum_{j \in M} \max_i b_{i,j} = \sum_{i \in \mathcal{N}} \sum_{j \in X_i} b_{i,j} \leq \sum_{i \in \mathcal{N}} v_i(X_i) = SW_v(b) \quad .$$

This gives us

$$SW_v(b) \geq OPT(v) - SW_v(b) \quad ,$$

which implies our claim. □

Of course, there is also a generalization of this proof to other equilibrium concepts and a more general form of smoothness called *weak smoothness.*

References and Further Reading

- Allan Borodin and Brendan Lucier. Price of Anarchy of Greedy Auctions. SODA'10. (The PoA result for greedy multi-minded CAs, results for general greedy algorithms)