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Due date: In the lecture on 8th May 2019, 12:15

Algorithmic Game Theory and the Internet

Summer Term 2019

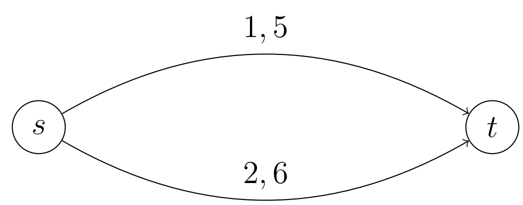
Exercise Set 5

Exercise 1: (1+3+2 Points)

Referring to the price of anarchy from Lecture 8 we can introduce a more optimistic point of view called the *price of stability*. For an equilibrium concept Eq, it is defined as

$$PoS_{\mathsf{Eq}} = \frac{\min_{p \in \mathsf{Eq}} SC(p)}{\min_{s \in S} SC(s)} \quad .$$

Consider the following symmetric network congestion game with two players:



(a) What is the Price of Anarchy and the Price of Stability of pure Nash equilibria?

(b) What is the Price of Anarchy and the Price of Stability of mixed Nash equilibria?

Hint: First of all, determine all mixed Nash equilibria. For this purpose, start with a sentence like "Let σ be a mixed Nash equilibrium with $\sigma_1 = (\lambda_1, 1 - \lambda_1)$, $\sigma_2 = (\lambda_2, 1 - \lambda_2)$" and subsequently derive properties of λ_1 and λ_2.

(c) What is the best upper bound for the Price of Anarchy that can be shown by smoothness?

Exercise 2: (3 Points)

State for each $M \geq 1$ a network congestion game with two players such that the Price of Anarchy of pure Nash equilibria is at least M.

Exercise 3: (3 Points)

A *fair cost-sharing game* is a congestion game such that for all resources $r \in \mathcal{R}$ the delay function can be modeled as $d_r(x) = c_r/x$ for a constant c_r. Show that fair cost sharing games with n players are $(n, 0)$-smooth.

Exercises 4 and 5 on the next page.

Exercise 4: (4 Points)

In the lecture, we assumed the social cost is given by $SC(s) = \sum_{i \in \mathcal{N}} c_i(s)$. On this basis, we defined the Price of Anarchy which will be denoted by $PoA^{\Sigma}_{\mathsf{Eq}}$. Another reasonable definition of the social cost could be $SC(s) = \max_{i \in \mathcal{N}} c_i(s)$. Hence, we get an additional definition of the Price of Anarchy $PoA^{\max}_{\mathsf{Eq}}$.

State an example of a game in which $PoA^{\Sigma}_{PNE} > PoA^{\max}_{PNE}$ and another game for $PoA^{\Sigma}_{PNE} < PoA^{\max}_{PNE}$.

Exercise 5: (4 Points)

Consider a (λ, μ)-smooth game with N players and let $s^{(1)}, \ldots, s^{(T)}$ be a sequence of states such that the external regret of every player is at most $R^{(T)}$. Moreover, let s^* denote a state that minimizes the social cost. We want to upper bound the average social cost of the sequence of states. For this purpose, prove the following bound

$$\frac{1}{T} \sum_{t=1}^{T} SC(s^{(t)}) \leq \frac{N \cdot R^{(T)}}{(1-\mu)T} + \frac{\lambda}{1-\mu} SC(s^*) \, .$$