Thomas Kesselheim

October 23, 2018
Due: lecture on November 6, 2018

Algorithms and Uncertainty

Winter Semester 2018/19

Exercise Set 2

Exercise 1: (5 Points)
Consider the following randomized rounding for fractional Ski Rental. In step t, flip an independent biased coin: With probability $\frac{x^{(t)}-x^{(t-1)}}{1-x^{(t-1)}}$ buy the skis, otherwise rent them. Show that if the underlying fractional algorithm is α-competitive so is the randomized integral algorithm.

Exercise 2: (5 Points)
Our lower bound for Online Set Cover assumes that the algorithm is *lazy*. Show that this assumption is indeed without loss of generality.
Consider an arbitrary online algorithm ALG, which sometimes selects unnecessary or multiple sets. Construct an algorithm ALG′ that always selects only a single set and only if it is necessary such that $\text{cost}(\text{ALG}'(\sigma)) \leq \text{cost}(\text{ALG}(\sigma))$ for all σ.

Hint: Keep ALG running in the background.

Exercise 3: (4 Points)
We would like to show that online algorithms that know the input distribution in advance are always deterministic.
To this end, we use the notation of Lecture 5. Let X be a random variable with values in $\mathcal{X}$ and A be an *independent* random variables with values in $\mathcal{A}$. Show that there is a $a \in \mathcal{A}$ such that $\mathbf{E}[c(a, X)] \leq \mathbf{E}[c(A, X)]$.

Hint: Use the same techniques as in the proof of Yao's principle.

Exercise 4: (1+5 Points)
Consider the following online problem: We would like to buy a house. Each time we view a house, we get to know how good it is. Immediately after, we have to decide whether to buy the current house or now. If we do not buy it now, there will be no other opportunity later. Formally, this means that a sequence of values is revealed one after the other. The algorithm can always decide if it stops the sequence at its current value or it it waits. We say that an algorithm wins if it stops the sequence at its highest value.

(a) Assume that the length of the sequence n is known. Give a randomized online algorithm that wins with probability $\frac{1}{n}$.

    Hint: Trivial.

(b) Use Yao's principle to show that no algorithm wins with higher probability.

    Hint: Use the profit-maximization variant of Yao's principle. The profit is 1 if the algorithm wins, 0 otherwise. The offline optimum is clearly always 1. Construct a probability distribution over sequences of the from $1, 2, \ldots, t, 0, \ldots, 0$.