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November 26, 2020
Due date: December 2, 2020 at noon

Algorithmic Game Theory

Winter Term 2020/21

Exercise Set 4

Exercise 1: (4+4 Points)
Consider the following regret-minimization-algorithm.

Greedy

- Set $p_1^1 = 1$ and $p_j^1 = 0$ for all $j \neq 1$.
- In each round $t = 1, \ldots, T$:
  Let $L_{min}^t = \min_{i \in N} L_i^t$ and $S^t = \{i \in N \mid L_i^t = L_{min}^t\}$.
  Set $p_i^{t+1} = 1$ for $i = \min S^t$ and $p_j^{t+1} = 0$ otherwise.

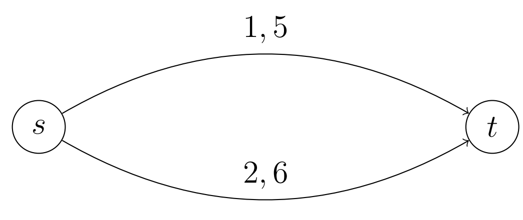
(a) Show that the costs of Greedy are at most $N \cdot L_{min}^T + (N-1)$.

(b) State a scheme for an example such that the stated upper bound of (a) is tight for an infinite number of values T.

Exercise 2: (1+3+2 Points)
Referring to the price of anarchy from Lecture 8 we can introduce a more optimistic point of view called the *price of stability*. For an equilibrium concept Eq, it is defined as

$$PoS_{\mathsf{Eq}} = \frac{\min_{p \in \mathsf{Eq}} SC(p)}{\min_{s \in S} SC(s)} \quad .$$

Consider the following symmetric network congestion game with two players:



(a) What is the Price of Anarchy and the Price of Stability of pure Nash equilibria?

(b) What is the Price of Anarchy and the Price of Stability of mixed Nash equilibria?

Hint: First of all, determine all mixed Nash equilibria. You might start with a sentence like "Let σ be a mixed Nash equilibrium with $\sigma_1 = (\lambda_1, 1-\lambda_1)$, $\sigma_2 = (\lambda_2, 1-\lambda_2)$" and subsequently derive properties of λ_1 and λ_2.

(c) What is the best upper bound for the Price of Anarchy that can be shown via smoothness?

Exercise 3: (4 Points)

Consider a (λ, μ)-smooth game with N players and let $s^{(1)}, \dots, s^{(T)}$ be a sequence of states such that the external regret of every player is at most $R^{(T)}$. Moreover, let s^* denote a state that minimizes the social cost. We want to upper bound the average social cost of the sequence of states. To this end, show the following bound

$$\frac{1}{T}\sum_{t=1}^{T} SC(s^{(t)}) \leq \frac{N \cdot R^{(T)}}{(1-\mu)T} + \frac{\lambda}{1-\mu} SC(s^*) \, .$$