

## Simple Mechanisms for Combinatorial Auctions

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Today, we continue our discussion of simple, non-truthful mechanisms. We consider combinatorial auctions, so there are  $m$  items  $M$ , which can each be allocated at most once. Bidders have valuation functions  $v_i: 2^M \rightarrow \mathbb{R}_{\geq 0}$ .

A *mechanism*  $\mathcal{M} = (f, p)$  defines a set of bids  $B_i$  for each player  $i \in N$  and consists of an *outcome rule*  $f: B \rightarrow X$ , where  $B = B_1 \times B_2 \times \dots \times B_n$ , and a *payment rule*  $p: B \rightarrow \mathbb{R}_{\geq 0}^n$ .

Last time, we introduced the definition of a smooth mechanism.

**Definition 16.1** (Smooth Mechanism, simplified version). *Let  $\lambda, \mu \geq 0$ . A mechanism  $\mathcal{M}$  is  $(\lambda, \mu)$ -smooth if for any valuation profile  $v \in V$  for each player  $i \in N$  there exists a bid  $b_i^*$  such that for any profile of bids  $b \in B$  we have*

$$\sum_{i \in N} u_i(b_i^*, b_{-i}) \geq \lambda \cdot \text{OPT}(v) - \mu \sum_{i \in N} p_i(b) .$$

It is easy to see that  $(\lambda, \mu)$ -smoothness implies that the Price of Anarchy for pure Nash equilibria is at most  $\frac{\max\{\mu, 1\}}{\lambda}$ . This proof also generalizes to (coarse) correlated equilibria. In a more complex argument, we were also able to show that the bound also holds for Bayes-Nash equilibria. Given these results, it is enough to show smoothness of mechanisms to bound the Price of Anarchy for all equilibrium concepts that we introduced so far. Interestingly, all results that we cover today were discovered before the smoothness framework, but the basic arguments were already present in the original publications.

## 1 Item Bidding

We first consider a truly simple, indirect mechanism. Instead of reporting complex functions  $2^M \rightarrow \mathbb{R}_{\geq 0}$ , the bidders now simply report a single bid  $b_{i,j}$  for each item  $j$ . Each item is sold in a separate first-price or second price-auction. That is, item  $j$  is assigned to the bidder  $i$  with the highest bid  $b_{i,j}$ . He has to pay  $b_{i,j}$ .

A bidder can potentially win multiple items, even if he only wants one. Recall *unit-demand valuations*: These are functions  $v_i$  such that there are  $v_{i,j} \in \mathbb{R}_{\geq 0}$  such that  $v_i(S) = \max_{j \in S} v_{i,j}$ . If, for example,  $v_{i,1} = \dots = v_{i,m} = 1$ , then bidder  $i$  has a value of 1 as long as he receives an item, no matter which. There is no way to express this in a bid. Therefore, this is not a direct mechanism and it cannot be truthful. However, its Price of Anarchy is bounded by 2.

**Theorem 16.2.** *For unit-demand valuations, item bidding with first-price payments is  $(\frac{1}{2}, 1)$ -smooth.*

*Proof.* We have to devise the deviation bids  $b_i^*$  for all bidders. These bids may depend on the valuations  $v$  but not on the bids. Consider the welfare-maximizing allocation on  $v$ . Let  $j_i$  be the item that is assigned to bidder  $i$  in this allocation. If  $i$  does not get any item, set  $j_i$  to  $\perp$ .

We now set  $b_{i,j}^* = \frac{v_{i,j}}{2}$  if  $j = j_i$  and 0 otherwise. That is, in the deviation bid, each bidder bids half his value on the item that he is supposed to get.

Given any bid profile  $b$ , bidder  $i$ 's utility after deviating is  $\frac{v_{i,j_i}}{2}$  unless another bidder bids at least  $\frac{v_{i,j_i}}{2}$  for item  $j_i$  in  $b$ . Therefore

$$u_i((b_i^*, b_{-i}), v_i) \geq \frac{v_{i,j_i}}{2} - \max_{i' \neq i} b_{i',j_i} \geq \frac{v_{i,j_i}}{2} - \max_{i'} b_{i',j_i} .$$

If we take the sum over all bidders  $i$ , then

$$\sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \sum_{i \in N} \frac{v_{i,j_i}}{2} - \sum_{i \in N} \max_{i'} b_{i',j_i} .$$

Observe that  $\sum_{i \in N} v_{i,j_i} = OPT(v)$  because of the way we defined  $j_i$ . Furthermore, we have  $\sum_{i \in N} \max_{i'} b_{i',j_i} \leq \sum_{j \in M} \max_{i'} b_{i',j} = \sum_{i \in N} p_i(b)$  because every item is counted at most once: For each item  $j$  there is at most one  $i$  such that  $j = j_i$ . That is,

$$\sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \frac{1}{2} OPT(v) - \sum_{i \in N} p_i(b) ,$$

which is exactly  $(\frac{1}{2}, 1)$ -smoothness.  $\square$

So, immediately we get that the Price of Anarchy for pure Nash equilibria is at most 2.

## 2 A Greedy Mechanism

Instead of selling items individually, one can also apply a smarter allocation algorithm and use a direct mechanism. We will now consider a mechanism based on the Greedy-by-Sqrt-Value-Density algorithm for combinatorial auctions. We introduced it as an algorithm for single-minded bidders. That is, each bidder is only interested in a single set of items. Under these circumstances, it can be turned into a truthful mechanism. Beyond this single-parameter domain, it cannot be turned into a truthful mechanism. However, as we will show, it can be turned into a mechanism of reasonable Price of Anarchy.

We assume that bidders report functions  $b_i: 2^M \rightarrow \mathbb{R}_{\geq 0}$ . (To ensure polynomial running time, only a polynomial number of bundles should have a positive value.) On the pairs  $(i, S)$  we run the greedy allocation rule. Each bidder gets only one such bundle  $S$ . If the mechanism wanted to allocate not only  $S$  to  $i$  but also  $S'$ , it would have to select the pair  $(i, S \cup S')$ .

By a simple extension of our analysis for single-minded bidders, one can show that the computed allocation is a  $\sqrt{2m}$ -approximation of the optimal declared welfare. However, it cannot be turned into a truthful mechanism as we showed even for single-minded valuations. Therefore, we build a very simple non-truthful mechanism. We combine the algorithm with a first-price payment rule: If bidder  $i$  gets set  $S$ , then his payment is exactly his bid on this set  $b_i(S)$ .

### First-Price Greedy Mechanism for Combinatorial Auctions

1. Collect bids  $b$ .
2. Sort the player-bundle pairs  $(i, S)$  by non-increasing score  $\frac{b_i(S)}{\sqrt{|S|}}$ .
3. Go through the sorted list and assign  $S$  to player  $i$  unless
  - (a) player  $i$  has already been allocated a bundle or
  - (b) one or more of the items in  $S$  has already been allocated.
4. Charge each player  $i$  his bid  $b_i(S)$  on the bundle  $S$  he is allocated.

**Theorem 16.3** (Borodin and Lucier, 2010). *The first-price greedy mechanism for multi-minded CAs is  $(1/2, \sqrt{2m})$ -smooth.*

*Proof.* Let  $(X_1^*, \dots, X_n^*)$  be an allocation that maximizes social welfare. That is,  $OPT(v) = \sum_{i \in N} v_i(X_i^*)$ . For each player  $i \in N$  let  $b_i^*$  be the single-minded declaration for set  $X_i^*$  at value  $v_i(X_i^*)/2$ . So, by bidding  $b_i^*$ , bidder  $i$  only tries to win the set that he is allocated in the social optimum.

Consider an arbitrary bid profile  $b$ . We know that the algorithm is monotone on single-minded bids. That is, if bidder  $i$  reports that he is only interested in set  $S$ , then there is a smallest bid with which player  $i$  wins bundle  $S$  against bids  $b_{-i}$ . Call this the *critical bid*  $\tau_i(S, b_{-i})$ .

In particular, bidding  $b_i^*$  against  $b_{-i}$ , bidder  $i$  may or may not win the set  $X_i^*$ . If he wins then  $u_i((b_i^*, b_{-i}), v_i) = v_i(X_i^*) - v_i(X_i^*)/2 = v_i(X_i^*)/2$ . If he loses, then the critical bid is at least  $v_i(X_i^*)/2$ . So in either case,

$$u_i((b_i^*, b_{-i}), v_i) \geq \frac{1}{2}v_i(X_i^*) - \tau_i(X_i^*, b_{-i}) .$$

Summing over all players  $i \in N$  we obtain

$$\sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) \geq \sum_{i \in N} \left( \frac{v_i(X_i^*)}{2} - \tau_i(X_i^*, b_{-i}) \right) = \frac{1}{2} \cdot OPT(v) - \sum_{i \in N} \tau_i(X_i^*, b_{-i}) .$$

Below, we will show the following lemma.

**Lemma 16.4.** *Fix bids  $b \in B$ . Let  $f(b)$  be the allocation chosen by the greedy mechanism for bids  $b$  and let  $X^*$  be another feasible allocation. Then,*

$$\sum_{i \in N} \tau_i(X_i^*, b_{-i}) \leq \sqrt{2m} \sum_{i \in N} b_i(f_i(b)) .$$

Once we have this lemma, we get

$$\begin{aligned} \sum_{i \in N} u_i((b_i^*, b_{-i}), v_i) &\geq \frac{1}{2} \cdot OPT(v) - \sqrt{2m} \cdot \sum_{i \in N} b_i(f_i(b)) \\ &= \frac{1}{2} \cdot OPT(v) - \sqrt{2m} \cdot \sum_{i \in N} p_i(b) , \end{aligned}$$

where the last step uses that the mechanism is a first-price mechanism. □

Note that apart from Lemma 16.4 this proof is actually pretty generic. It looks exactly like the smoothness proof for a first-price auction and uses hardly any property of the mechanism. It still remains to prove Lemma 16.4, which indeed relies on the mechanism using a greedy rule.

*Proof of Lemma 16.4.* Let  $\epsilon > 0$ . For all  $i$ , let  $b_i^*$  be the single-minded declaration for set  $X_i^*$  at value  $\tau_i(X_i^*, b_{-i}) - \epsilon$ . Let  $b'_i$  be the point-wise maximum of  $b_i$  and  $b_i^*$ . A crucial property of the greedy algorithm is that the allocation it chooses on profile  $b'$  is the same as on  $b$ . The reason is that all introduced new bids are below the respective critical bids. Some pairs  $(i, S)$  move towards the front in the sorted list. However, none of them moves beyond the point at which it gets accepted. So, its presence does not have any influence of the algorithm. So, formally,  $f(b) = f(b')$ . Besides, if  $b_i(S) \neq b'_i(S)$  for a set  $S$ , then bidder  $i$  does not get set  $S$  in  $f(b)$  or  $f(b')$ .

That is,

$$\sum_{i \in N} b_i(f_i(b)) = \sum_{i \in N} b_i(f_i(b')) = \sum_{i \in N} b'_i(f_i(b')) .$$

Now we use the fact that the algorithm is a  $\sqrt{2m}$ -approximation. As  $X^*$  is a feasible allocation, we have

$$\sum_{i \in N} b'_i(f_i(b')) \geq \frac{1}{\sqrt{2m}} \sum_{i \in N} b'_i(X_i^*) .$$

By definition of  $b'_i$ , we also have

$$\sum_{i \in N} b'_i(X_i^*) = \sum_{i \in N} \max \{ b_i(X_i^*), \tau_i(X_i^*, b_{-i}) - \epsilon \} \geq \sum_{i \in N} (\tau_i(X_i^*, b_{-i}) - \epsilon) = \sum_{i \in N} \tau_i(X_i^*, b_{-i}) - n\epsilon .$$

So, in combination

$$\sum_{i \in N} b_i(f_i(b)) \geq \frac{1}{\sqrt{2m}} \sum_{i \in N} \tau_i(X_i^*, b_{-i}) - n\epsilon .$$

This holds for all  $\epsilon > 0$ . The claim follows by taking the limit as  $\epsilon \rightarrow 0$ .  $\square$

### 3 Second-Price Auctions

Our results today were for generalizations of the first-price auction. Maybe it would be more natural to generalize the second-price auction. In the case of item bidding this would mean that each item is sold in a separate second-item auction. For the greedy mechanism, we could charge every player the respective critical bid. However, there are some issues as we see in this example.

**Example 16.5.** *Consider a single-item second-price auction with two bidders. Let  $\epsilon > 0$  be small. Then for  $v_1 = 1$ ,  $v_2 = \epsilon$ , it is a pure Nash equilibrium  $b_1 = 0$ ,  $b_2 = 1$ . Here the second bidder pays nothing and wins the item. The first bidder does not want to bid more because they would have to pay at least 1 to win the item. So, the Price of Anarchy is unbounded.*

One can indeed get bounds on the Price of Anarchy when assuming that bidders do not overbid. See the referenced papers for more details.

### References and Further Reading

- George Christodoulou, Annamária Kovács, Michael Schapira: Bayesian Combinatorial Auctions. J. ACM 63(2): 11:1-11:19 (2016) (The first paper on item bidding.)
- Allan Borodin and Brendan Lucier. Price of Anarchy of Greedy Auctions. SODA'10. (The PoA result for greedy multi-minded CAs, results for general greedy algorithms)