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Due date: October 20, 2021 at 10:00 a.m.

Algorithmic Game Theory

Winter Term 2021/22

Exercise Set 1

If you want to hand in your solutions for this problem set, please send them via email to alexander.braun@uni-bonn.de – *make sure to send a pdf-file which contains your name and your email address. Of course, submitting solutions in groups is also possible.*

If you would like to present one of the solutions in class, please also send an email to alexander.braun@uni-bonn.de *containing the **task** which you would like to present and in **which of the tutorials** you would like to do so. Deadline for the email is Wednesday, 10:00 am. Please note that the tasks will be allocated via a first-come-first-served procedure, so sending this email earlier than Wednesday morning is highly recommended.*

Exercise 1: (6 Points)

Give an example for a symmetric network congestion game (strategies are s-t paths in a directed graph with the same s and t for all players) with monotonically increasing delay functions d_r such that there exist at least two pure Nash equilibria with different *social costs*. We define the social cost to be the sum of all players' costs $\sum_{i \in \mathcal{N}} c_i(S)$.

Exercise 2: (5+1 Points)

In a *weighted* congestion game, every player $i \in \mathcal{N}$ has an individual weight $w_i > 0$. The delay of a resource r now depends on the sum of the weights – instead of the number of players – of those players who are using r. For this purpose, we could redefine $n_r(S)$ to be $n_r(S) = \sum_{i : r \in S_i} w_i$ and consider delay functions $d_r \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$. Like in the unweighted case the cost of player i is defined as $c_i(S) = \sum_{r \in S_i} d_r(n_r(S))$.

(a) Prove that weighted congestion games do not fulfil the *Finite-Improvement Property*, even having only two players, three resources and two strategies for each player.

Hint: Consider $\mathcal{N} = \{1, 2\}$, $w_1 = 1$, $w_2 = 2$, $R = \{a, b, c\}$, $\Sigma_1 = \{\{a\}, \{b, c\}\}$, $\Sigma_2 = \{\{b\}, \{a, c\}\}$. Choose delay functions such that

$$(\{a\}, \{b\}) \to (\{a\}, \{a, c\}) \to (\{b, c\}, \{a, c\}) \to (\{b, c\}, \{b\}) \to (\{a\}, \{b\})$$

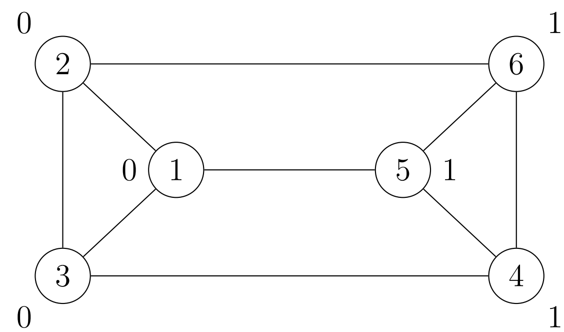
is a sequence of best response improvement steps.

(b) Use part (a) to show that a pure Nash equilibrium does not need to exist.

Exercise 3: (1+3+2 Points)

In a *consensus game*, we are given an undirected graph $G = (V, E)$ with vertex set $V = \{1, \dots, n\}$. Each vertex $i \in V$ is a player and her action consists of choosing a bit $b_i \in \{0, 1\}$. Let $N(i) = \{j \in V \mid \{i, j\} \in E\}$ denote the set of neighbours of player i, i.e., all players j connected to i via an edge. Furthermore, let $\vec{b} = (b_1, \dots, b_n)$ be the vector of players' choices. The loss $D_i(\vec{b})$ for player i is the number of neighbours that she disagrees with, i.e.,

$$D_i(\vec{b}) = \sum_{j \in N(i)} |b_j - b_i| \, .$$



a) Calculate the loss D_i of player 1 for the actions depicted in the graph above.

b) Show that a consensus game represented as an undirected Graph G can also be modeled as a congestion game Γ. To this end, specify the tuple $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ and show that the loss D_i coincides with the cost c_i.

c) Prove that in a congestion game modeling a consensus game with $|V| = n$ players all improvement sequences have length $O(n^2)$.