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Due date: November 17, 2021 at 10:00 a.m.

Algorithmic Game Theory

Winter Term 2021/22

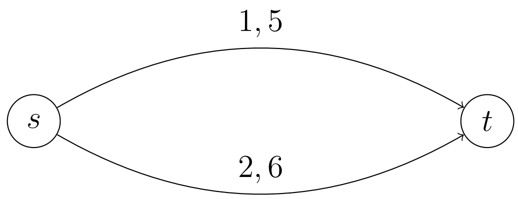
Exercise Set 5

Exercise 1: (1+3+2 Points)

Referring to the Price of Anarchy from Lecture 8, we introduced a more optimistic point of view called the *Price of Stability* in Lecture 9. For an equilibrium concept Eq, it is defined as

$$PoS_{\mathsf{Eq}} = \frac{\min_{p \in \mathsf{Eq}} SC(p)}{\min_{s \in S} SC(s)} \quad .$$

Consider the following symmetric network congestion game with two players:



(a) What is the Price of Anarchy and the Price of Stability of pure Nash equilibria?

(b) What is the Price of Anarchy and the Price of Stability of mixed Nash equilibria?

Hint: First of all, determine all mixed Nash equilibria. You might start with a sentence like "Let σ be a mixed Nash equilibrium with $\sigma_1 = (\lambda_1, 1 - \lambda_1)$, $\sigma_2 = (\lambda_2, 1 - \lambda_2)$" and subsequently derive properties of λ_1 and λ_2.

(c) What is the best upper bound for the Price of Anarchy that can be shown via smoothness?

Exercise 2: (4 Points)

Consider a (λ, μ)-smooth game with N players and let $s^{(1)}, \dots, s^{(T)}$ be a sequence of states such that the external regret of every player is at most $R^{(T)}$. Moreover, let s^* denote a state that minimizes the social cost. We want to upper bound the average social cost of the sequence of states. To this end, show the following bound:

$$\frac{1}{T} \sum_{t=1}^{T} SC(s^{(t)}) \leq \frac{N \cdot R^{(T)}}{(1-\mu)T} + \frac{\lambda}{1-\mu} SC(s^*) \, .$$

Exercise 3: (3+2 Points)

We call s an ϵ-approximation to a pure Nash equilibrium if $c_i(s) \leq (1+\epsilon) c_i(s'_i, s_{-i})$ for all i and s'_i.

(a) Consider a (λ, μ)-smooth cost-minimization game and let and $0 < \epsilon < \frac{1}{\mu} - 1$. Prove that the PoA of ϵ-approximations to pure Nash equilibria is at most $\frac{(1+\epsilon)\lambda}{1-(1+\epsilon)\mu}$.

(b) Can you state a similar result for more general equilibrium concepts?