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Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

Lemma 66 *If $v > v_c$ then function $F(Z)$ has an infinite, discrete set of complex poles none of which are real.*

We are going to apply the following result from complex function theory; see, for example, Flajolet (2009).

Theorem 67 (Pringsheim) *Let $H(Z) = \sum_{n=0}^{\infty} a_n Z^n$ be a power series with finite radius of convergence, R. If $H(Z)$ has only non-negative coefficients a_n, then point $Z = R$ is a singularity of $H(z)$.*

Now we are ready to prove Theorem 59.

Proof.[Proof of Theorem 59] Suppose that the fighter's speed v is larger than $v_c \approx 2.6144$. By Lemma 66, $F(Z)$ does have a discrete set of poles, and therefore, a finite radius of convergence, R. If all coefficients F_j of $F(Z)$ were positive, R would be a singularity of $F(Z)$, by Pringsheim's theorem; but we know from Lemma 66 that there are no real singularities. Thus, there must be coefficients $F_j \leq 0$, and we conclude from Lemma 64 that the fighter succeeds in containing the fire. □

5.7 General Constructions: Lower and upper bounds

Now we consider the case that the firefighter has some speed v but is able to distribute the speed for the construction of more than one firebreak at the same time. In some sense this would mean that the firefighter are also able to jump from one position to the other. The firebreak can be constructed everywhere outside the spreading fire as long as the overall speed is not exceeded.

Due to our considerations above there is a simple upper bound $v > 2$ for a spreading fire circle. We simply construct to symmetric logarithmic spirals along the boundary of the fire with excentricity $v/2 = \frac{1}{\cos\alpha}$ for $v/2 > 1$ and guarantee to enclose the fire in any case as depicted in Figure 5.22.

There is a lower bound of $v \geq 1$ which already makes some effort as we will see below, For the interval $v \in (1, 2)$ it is still an open question whether there is a strategy for v or not.

Theorem 68 *For any speed $v > 2$ there is a successful general strategy that encloses any spreading fire circle. For speed $v \leq 1$ there is no such general strategy.*

Proof. The upper bound was shown above as depicted in Figure 5.22. For the lower bound we choose $v = 1$ and assume that there is a successful strategy S. The strategy S will finally enclose the spreading fire. Let x denote the final *stone* for the enclosement at some time t_x. There is a situation as depicted in Figure 5.23. There will be some outer connected boundary firebreak S_O which consists of a *ring* R_O and some simple tree paths S_O^j which are connected with R_O. Other tree like connected paths, S_I^i, constructed by S are not connected to the outer boundary. They serve as additional obstacles for slowing down the spread of the fire. We can assume that finally at time t_x the strategy builds the first overall loop. So all S_I^i and S_O^j are tree like paths.

Let Π_s^x denote the geodesic shortest path from the source of the fire to the point x under the presence of the obstacles R_O and S_O. Assume that this path crosses n inner paths $S_I^{j_i}$ for some subset $\{j_1, j_2, \ldots, j_n\}$. For the contradiction we can omit these obstacles by the following argument. Any connected tree like obstacle path $S_I^{j_i}$ that intersects with Π_s^x has a starting point s_{i_j} and a leaving point t_{i_j} w.r.t. the orientation of Π_s^x from s to x. Following the boundary from s_{i_j} to t_{i_j} in clockwise and conuterclockwise direction along $S_I^{j_i}$ gives two path with length twice

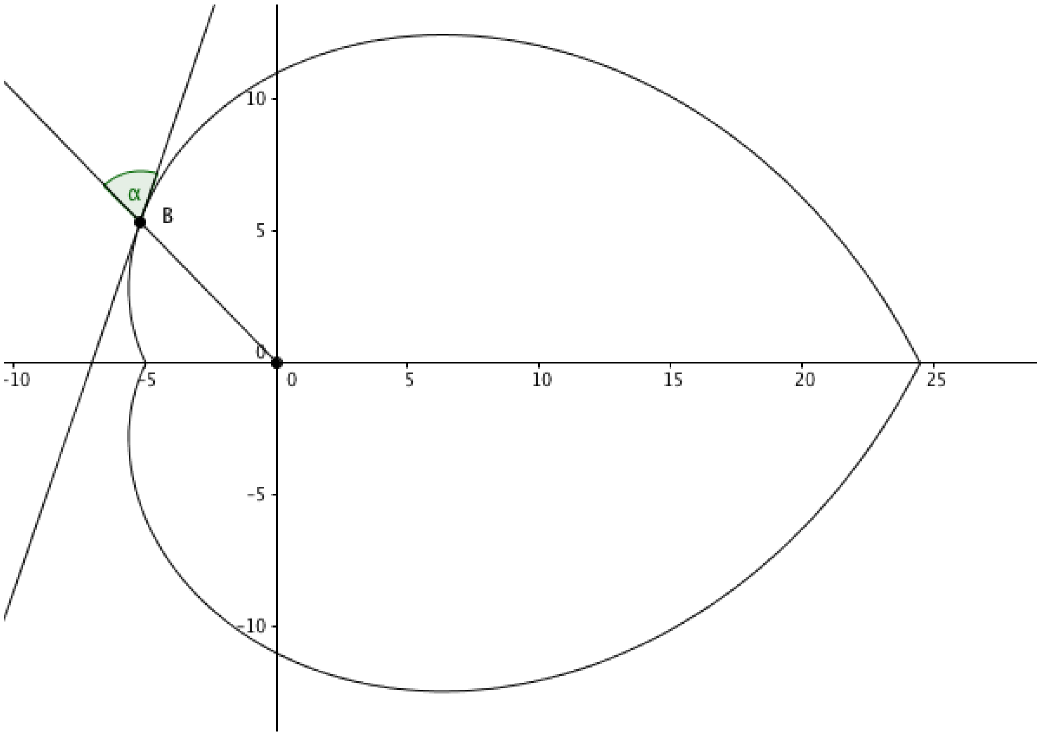

Figure 5.22: For $v > 2$ the firefighter simultaneously build two spiral of excentricity α for $v/2 = \frac{1}{\cos\alpha}$ at the boundary of the fire. The strategy is successful for any such speed $v > 2$.

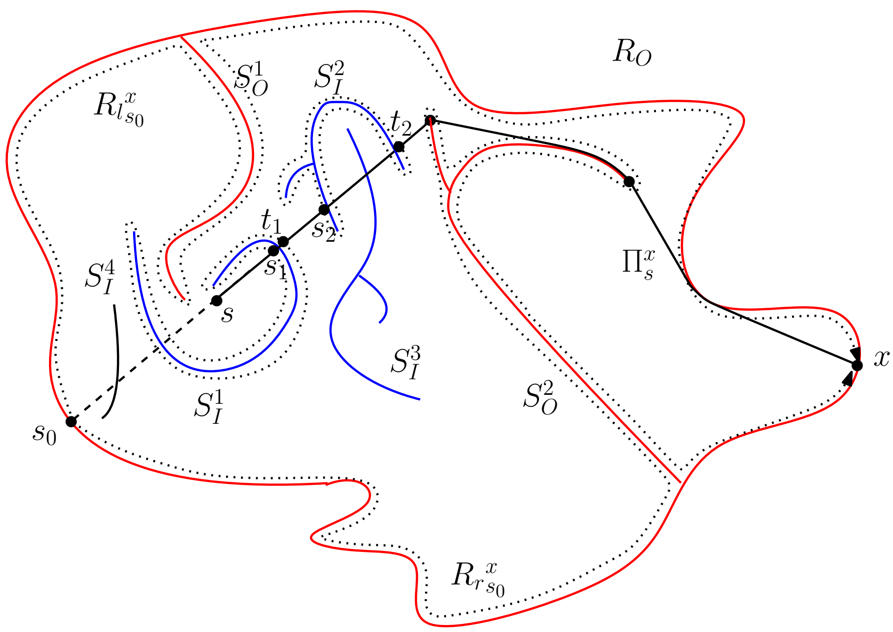

Figure 5.23: Asume that for $v = 1$ the spreading fire starting from s is finally enclosed at x at time t_x. The inner obstacles $S_I^{j_i}$ can be omitted. The fire will reach the final *stone* x along Π_s^x before all obstacles can be constructed. This means that there is no such construction!

the length of $S_I^{j_i}$, the shorter of these two path has lenght not larger than $S_I^{j_i}$. Therefore, during the construction of $S_I^{j_i}$ the fire can move form s_{i_j} to t_{i_j}. This holds because the construction was done with the same speed $v = 1$. By this argument we simply omit all inner obstacles $S_I^{j_i}$. We prolong the path Π_s^x back to the outer boundary to some point s_0 on R_O and S_O as depicted in Figure 5.23 such that $\Pi_{s_0}^x$ is a shortest path from s_0 to x under the presence of R_O and S_O. The sum of the path length of the outer boundary exceeds the length of $\Pi_{s_0}^x$, because in the worst case the path $\Pi_{s_0}^x$ can only follow the full paths R_O and S_O. Thus, before x can be placed the fire has reached the point x along the path $\Pi_{s_0}^x$ which is a bit shorter than $\Pi_{s_0}^x$. The outer boundary cannot be closed at point x. □

Exercise 29 *Show that in the above lower bound construction it is allowed that we can assume that no* loop obstacles *(apart from R_O) have been constructed.*

The above proof follows an idea of Bresson et al. They brought up the question whether it is possible to show that the lower bound lies inside $v \in (1, 2)$ or is indeed $v = 2$. This is still an open question.

Exercise 30 *Show that in the above proof there is not much room for improving the lower bound to $v > 1$. Try to construct a corresponding example.*