Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16

Introduction

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Organisation

- Lecture: Tuesday 16:15 to 17:45
- Exercise groups: Starting next week 28/29th
  Wednesday: 14-16
  Thursday: 10-12
- Sign in
- Manuscipt on the webpage
- Slides on the webpage
- Exercises
- Today: Introduction, different topics

Main problems and intention

- Evader/Intruder versus Searcher/Guard
- Escaping/Intruding versus Catching/Avoidance
- Game, Competition
- Different Scenarios: Environment, Facilities, Goal, Model
- Discrete, Continuous, Geometry, Combinatorics
- Interpretation: Possible Position of the Intruder, Decontamination, Firefighting

Theoretical Aspects

- Algorithmic track
- Computational complexity
- Correctness or Failure
- Efficiency
- Optimality
- Prerequisites: Algorithms, Datastructure, Analysis, Complexity, Computability
- Models, Methods, Proof Techniques, Tools
- Today Introduction

- Continuous Problem
- Complexity Result
- NP-hardness
- Reduction

Optimal-Closing-Sequence:
Instance: Simple polygon, set of n intruders, set of m *doors* to be closed successively time c_i, safes area A_i.
Output: Compute the optimal sequence of doors that has to be closed for maximizing the are the area safed.

Reduction: Subset-Sum with treshhold

Subset-Sum:
Instance: n integer numbers $a_1, a_2, \ldots, a_n$, integer treshhold t
Output: Sum of a subset of $a_1, a_2, \ldots, a_n$ as close as possible to t, not exceeding t.

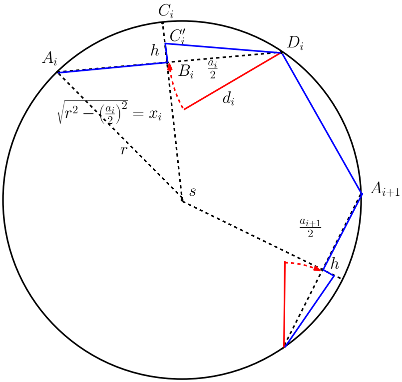
- Reduction to *Optimal-Closing-Sequence*
- Construct Instance in polynomial time
- Solution for *Optimal-Closing-Sequence* $\Leftrightarrow$ Solution for *Subset-Sum*

Reduction: Subset-Sum with treshhold

- Circle radius r, center s, intruder start at s
- Chords of length a_i, polygonal chain: A_i,B_i,C_i',D_i
- Door d_i safes Area $A_i = \frac{ha_i}{4}$
- Speed $v(t+0.5) = r$ for every Intruder
- Choose r so that $vt < x_i = \sqrt{r^2 - \left(\frac{a_i}{2}\right)^2}$
- Substituting v by $\frac{r}{(t+0.5)}$:

$$\left(\frac{a_i}{2}\right)^2 < \left(1 - \frac{t^2}{(t+0.5)^2}\right) r^2$$

Reduction: Subset-Sum with treshhold



Reach C_i' after $t + 0.5$ steps, do not reach B_i after t steps. Maximize!

Reduction: Subset-Sum with treshhold

Theorem 1: Computing an optimal-enclosement-sequence is NP-hard.

Proof: Reduction from Subset-Sum, Equivalence!

Example II: Grid Graph

- Discrete Problem
- Correctness/Failure
- Structural Properties

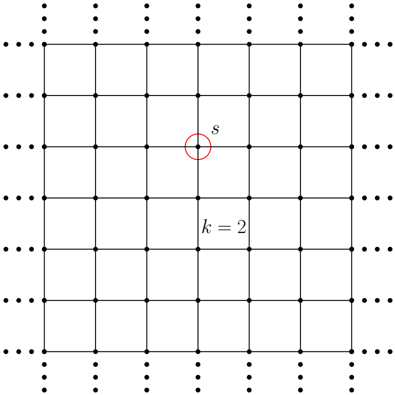
Evader-Enclosement in Grid-Graphs

Instance: A rectangular grid, a start vertex s of the evader and k protecting guards per time step.
Output: Compute an efficient protection strategy that encloses the evader (and finally find the evader).

A Two Player Game!

Example II: Grid Graph, $k = 2$

Evader moves (4Neighborship), Guards will be placed



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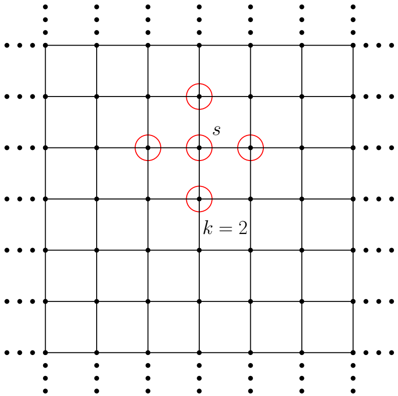

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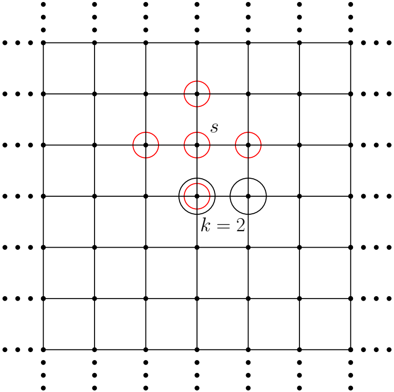

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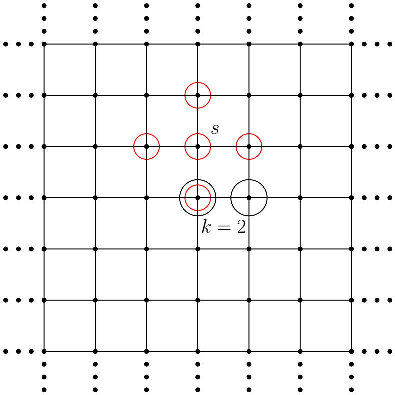
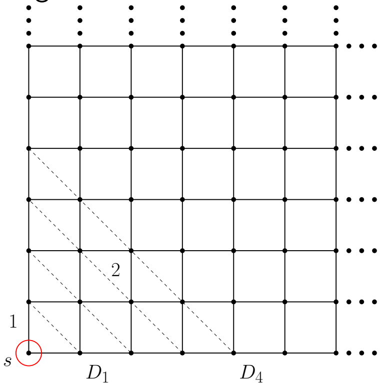


Example Applet! Enclosing the Evader first!

Lemma 2: Catching an evader in a grid world by setting $k = 1$ blocking cells after each movement of the evader cannot succeed in general.



Step l: r_l blocked cells in $D_{l+1}, D_{l+2}, \ldots$
$B_l \subseteq D_l$ burning cells in D_l

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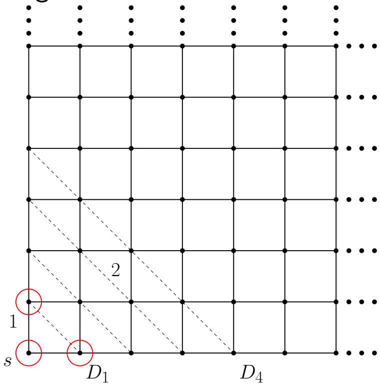

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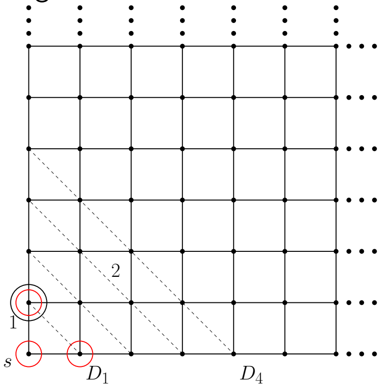

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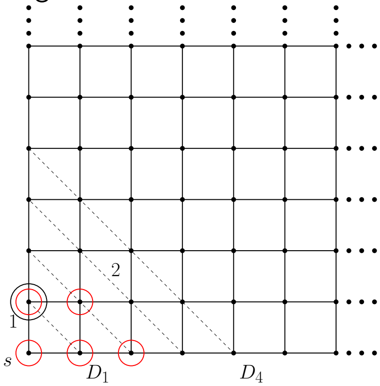

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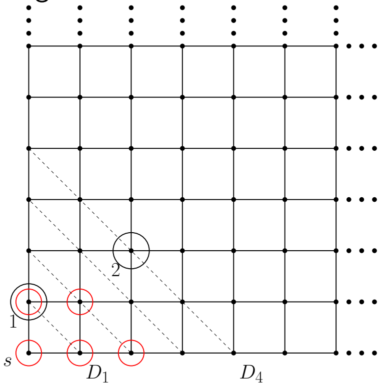

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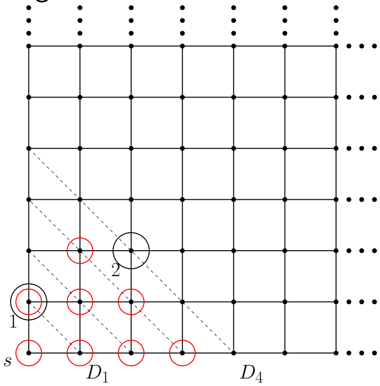

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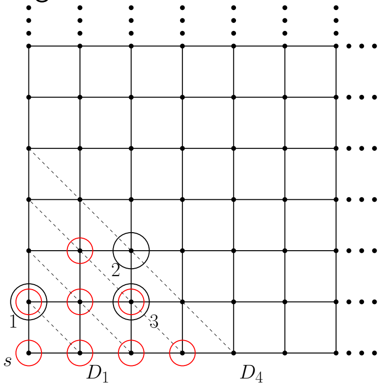

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- Block of the guard in D_{l_1}: $l_1 > l + 1$
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Example II: Grid Graph, $k = 2$

Lemma 3: For $k = 2$ there is a successful enclosement strategy, that encloses the evader after 8 steps. After 9 additional steps, the evader will be found.

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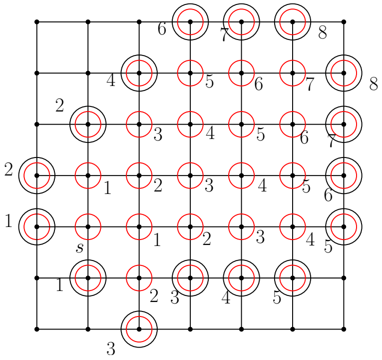

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Firefigthing interpretation! Outside the fire!

Lemma 3: For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successul strategy encloses an area of at least 18 burning vertices. This bound is tight.

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Example II: Grid Graph, $k = 2$

Firefigthing interpretation! Integer LP for $l \leq 8$, $T \leq 9$

$$\text{Min } \sum_{v \in L} b_{v,T}$$

$$
\begin{array}{rcll}
b_{v,t} + d_{v,t} - b_{w,t-1} & \geq & 0 & : \forall v \in L, v \in N(w), 1 \leq t \leq T \\
b_{v,t} + d_{v,t} & \leq & 1 & : \forall v \in L, 1 \leq t \leq T \\
b_{v,t} - b_{v,t-1} & \geq & 0 & : \forall v \in L, 1 \leq t \leq T \\
d_{v,t} - d_{v,t-1} & \geq & 0 & : \forall v \in L, 1 \leq t \leq T \\
\sum_{v \in L}(d_{v,t} - d_{v,t-1}) & \geq & 2 & : \forall 1 \leq t \leq T \\
b_{v,0} & = & 1 & : v \in L \text{ is the origin } (0,0) \\
b_{v,0} & = & 0 & : v \in L \text{ is not the origin } (0,0) \\
d_{v,0} & = & 0 & : \forall v \in L \\
d_{v,t}, b_{v,t} & \in & \{0,1\} & : \forall v \in L, 1 \leq t \leq T
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$$

Example II: Grid Graph, $k = 2$

Lemma 4: For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successul strategy encloses an area of at least 18 burning vertices. This bound is tight.

Optimal solution by LP solver:

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Lemma 4: For the outbreak of a fire on a single source in a grid and the usage of two firefighters per time step any successul strategy encloses an area of at least 18 burning vertices. This bound is tight.

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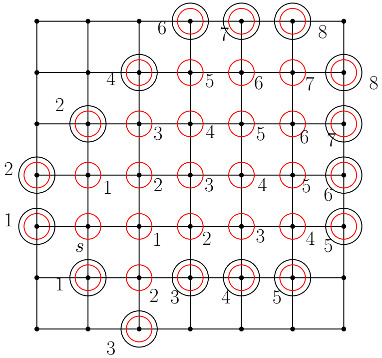

Example III: Continuous Firefigthing

Geometric Firefigther Problem
Instance: A circle with center C of radius A that grows with unit speed. An agent who builds a firebreak with speed $v > 1$
Output: Compute a strategy that finally fully enclose the spreading fire.

Example III: Continuous Firefigthing

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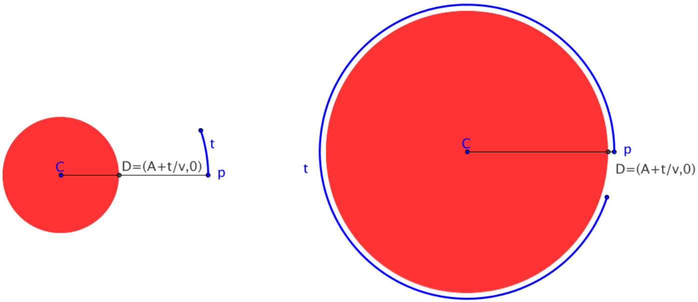
A circular strategy!



Lemma 5: Enclosing a fire of extension A with a single circular loop around the source of the fire is possible, if and only if the speed v of the firefigther is larger than 2π.

Proof:

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- Choose $p = (A + x, 0)$ away from the fire

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GeoGebra Simulation

Example IV: Firefigthing Grid-World Simulation

Discrete Firefigther Problem

Instance: Grid contamination of size B, spreads 4Neighborship after n time steps. Agent cleans a cell, builds a wall cell and leaves the cell within b time steps.

Output: Compute a strategy that finally fully enclose the spreading fire.

Example IV: Firefigthing Grid-World Simulation

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Example: $n = 30$, $b = 5$, $B = 3 \times 3$

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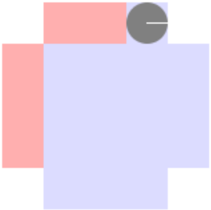

Example IV: Firefigthing Grid-World Simulation

Conjecture 1: For a grid fire thats spreads after n time steps and an agent that builds a wall within b time steps, the spiral strategy only succeeds if $b < \frac{n-1}{2}$ holds.

By simulation!