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Algorithms and Uncertainty

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Tutorial Session - Week 1

You are supposed to work on these tasks in class together with your fellow students. Therefore, you are sent into Zoom Breakout-Rooms together with 1-3 other students. Once entered, make sure your camera and microphone are switched on. If you do not know each other yet, each of you could start with a very quick introduction: What's your name? Do you study Computer Science or maybe something else (Maths, Economics,...)? Do you have any prior knowledge on Algorithms and Uncertainty (Online Algorithms, Markov Decision Processes, Learning Problems,...) already or is this your first course in this area?

Afterwards, you are supposed to discuss the exercises on this sheet. ***Note that you should see this also as a chance to talk about definitions, proof ideas and techniques in addition to only working out a formal solution for the tasks.*** *If you do not know a definition or theorem by hard, feel free to open the lecture notes and have a look. Further, if you have any questions, I will drop by in your Breakout-Room to discuss possible issues with you.*

If there is some time remaining at the end of the tutorial, all of us will meet again so that you can share your ideas on the tasks with the whole group.

Exercise 1:
We want to recall the basics of linear programming. Therefore, we consider the *Vertex Cover* problem: The task is to cover edges in a graph where an edge can be covered by its incident vertices. More formally, a vertex cover is a set of vertices $S \subseteq V$ such that for all $e = \{u, v\} \in E$ either $u \in S$ or $v \in S$. We are interested in finding a vertex cover of minimum size. For now, we restrict to bipartite graphs, i.e. graphs $G = (V, E)$ with $V = L \cup R$.

(a) Give the integer program of the Vertex Cover problem and its LP relaxation.

(b) Give the dual program to the LP from (a).

Additionally, consider the maximum matching problem in the bipartite version, i.e. given a bipartite graph $G = (V, E)$ with $V = L \cup R$ our goal is to compute a maximum matching where a matching is a set of edges $M \subseteq E$ such that no two edges in M share a common vertex.

(c) Compare the dual program from (b) to the LP relaxation of the maximum matching problem for bipartite graphs. Do you notice any similarities?

Exercise 2:

We consider the online maximum bipartite matching problem: Assume there is a bipartite graph $G = (L \cup R, E)$, but initially, we only know the set L. In each round, one vertex in R is revealed with all its incident edges. We now have to decide immediately and irrevocably if we want to select one of these edges or leave r unmatched. As in the offline matching problem, any vertex in L can only be matched to at most one vertex in R.

A very simple algorithm is the following one: Fix an ordering of vertices in L. Whenever a vertex $r \in R$ is revealed, if there is still an unmatched neighbor, match it to the unmatched neighbor of smallest index.

Show that $|\mathrm{ALG}| \geq \frac{1}{2}|\mathrm{OPT}|$ for any input graph G and any arrival order of vertices in R, where $|\mathrm{ALG}|$ denotes the size of the matching computed by the algorithm and $|\mathrm{OPT}|$ is the maximum offline matching in G.