MA-INF 1203 Discrete and Computational Geometry

Wintersemester 2019/20
Assignment 6

Deadline: **19** November before noon (To be discussed: **19/20**. November 2019)

1 Convex hulls of random point sets

We consider a set P of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in $[0,1]^2$, where the coordinates are chosen uniformly and independently at random from $[0,1]$.

(a) Consider a random assignment to $x_1, ..., x_n, y_1, ..., y_n$ as defined above. Assume that all x_i and y_i are different. Let $\sigma : [n] \to [n]$ be the permutation such that $x_{\sigma(i)} < x_{\sigma(j)}$ for all $i < j$. Derive an upper bound on the probability that the r-th point from the left, $(x_{\sigma(r)}, y_{\sigma(r)})$, lies on the convex hull.

(b) Show that the expected number of vertices of the convex hull of P is in $O(\log n)$.

Hint: You may use the fact that all sorted orders of the random variables x_i and y_i are equally likely. More precisely, let $\tau : [n] \to [n]$ be the permutation such that $y_{\sigma(\tau(i))} < y_{\sigma(\tau(j))}$ for all $i < j$. In words, τ^{-1} is the permutation that would turn the list of points, sorted by x-coordinates, into a list sorted by y-coordinates. You may use the fact that all permutations τ are equally likely, for any σ.

2 Faster computation of the convex hull

Consider a set $P \subset \mathbb{R}^3$ of n points. Assume that if we pick a random sample $Y \subset P$ of size r then the expected number of points in Y on the boundary of $\text{conv}(Y)$ is $O(r^\alpha)$, for some constant $\alpha < 1$. Prove that under this condition, the expected running time of the convex hull algorithm given in the lecture is in $O(n)$.

3 Delaunay triangulations and Voronoi diagrams

a) Define a graph on a set $P \subset \mathbb{R}^2$ as follows: Two points p and q are connected by an edge if and only if there exists a disk with both p and q on the boundary and with no point of P in its interior. Prove that this graph is the Delaunay triangulation of P.

b) Given a Delaunay triangulation of $P \subset \mathbb{R}^2$ as a doubly connected edge list (DCEL), compute the graph of the Voronoi Diagram of P. Use the definition of the Voronoi diagram that includes an additional vertex at infinity that is incident to all unbounded Voronoi edges. Use the following definition of a DCEL:

– The vertex record of a vertex v stores the coordinates of v in $Coordinates(v)$. It also stores a pointer $IncidentEdge(v)$ to an arbitrary half-edge that has v as its origin.

- The face record of a face f stores a pointer $IncidentEdge(f)$ to some half-edge on its boundary (this also holds for the outer face).
- The half-edge record of a half-edge $\vec{e}$ stores a pointer $Origin(\vec{e})$ to its origin, a pointer $Twin(\vec{e})$ to its twin half-edge, and a pointer $IncidentFace(\vec{e})$ to the face that it bounds. The origin is chosen such that $IncidentFace(\vec{e})$ lies to the left of $\vec{e}$ when it is traversed from origin to destination. The half-edge record also stores pointers $Next(\vec{e})$ and $Prev(\vec{e})$ to the next and previous edge on the boundary of $IncidentFace(\vec{e})$.

c) Show that the vertical projection of the edges of the polytope in $\mathbb{R}^3$ constructed via the lifting map as in the lecture is the Voronoi diagram of P.