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Theoretical Aspects of Intruder Search

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The manuscript will be successively extended during the lecture in the Wintersemester. Hints and comments for improvements can be given to Elmar Langetepe by E-Mail `elmar.langetepe@informatik.uni-bonn.de`. Thanks in advance!

Chapter 3

Discrete Cop and Robber game

In this chapter we would like to discuss another discrete variant of the intruder search problem. In comparison to the previous chapter, we assume that at any time of the game the position of the single intruder is given.

More precisely, there is a single robber R and a set of cops C and a graph $G = (V, E)$. The game starts with the cops, by choosing the starting vertices for the set C. After that, the robber R can choose its starting vertex. The game runs in subsequent steps. First, any cop can move from a vertex to an incident vertex, then the robber can move. The game ends, when one cop enters the position of the robber or the robber enters the position of a cop, respectively.

Cop and Robber game for graphs:
Instance: A Graph $G = (V, E)$ and the cardinality of the cops C.
Question: Is there a winning strategy S for the cops C?

We are searching for classifications of graphs that allow a winning strategy for C or vice versa a winning strategy for R. Aigner and Fromme introduced the problem in the midst of the 90ies.

3.1 Classifications of graphs

3.1.1 Simple examples and pitfalls

It is interesting to see that it makes a difference, if we do not allow the robber to keep in place during its strategy. This is called the *active* version of the game, in correspondance to the *passive* version, where the robber is not forced to move in any step.

Figure 3.1 shows an example where this makes a difference for a single cop. In the active version the cop starts at vertex v_1 and the robber can only choose the opposite vertex r_2. The cop moves toward v. Now the robber has to move to r_1. The cop moves toward v_2 and after the next mandatory move of the robber, the robber will be catched. In the passive version the robber can move around or rest in the 4-cycle and holds distance 2 from the cop all the time. In the following we will always discuss the more intuitive passive version of the game. Let G_C denote the set of all graphs that allow a winning strategy for C and let G_R denote all graphs that have a winning strategy for R.

Obviously, any tree T belongs to G_C already for a single cop, that successively moves into the subtree of R. Additionally, for a single cop, all graphs that contain a cycle of length at least 4 belong to G_R.

We concentrate on a single cop. In the winning case for the cop, the final situation is as follows: The robber is located in a vertex v_r and the cop is located in v_c for an edge $e = (v_r, v_c)$. Moreover, all neighbors, $N(v_r)$, of v_r are also neighbors of v_c, which means $N(v_r) \subseteq N(v_c)$.

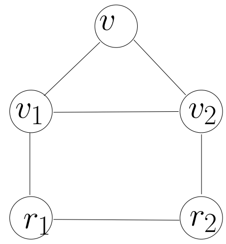

Figure 3.1: In this simple graph for one cop and a robber it makes a difference, if the robber has to perform moves mandatorily.

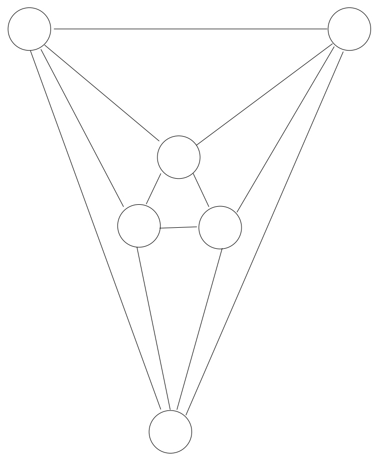

Figure 3.2: A graph without pitfalls.

For a pair (v_r, v_c) of vertices we call v_r a *pitfall* and v_c its *dominating vertex* if $N(v_r) \cup \{v_r\} \subseteq N(v_c)$ holds. Obviously, a graph G whithout a pitfall is in G_R. Figure 3.2 shows an example.

Exercise 15 *Present a construction scheme for graphs of arbitrary size without pitfalls.*

3.1.2 Algorithmic approaches

We would like to show that for a single cop the classification of a graph depends on the successive removement of pitfalls of G.

Lemma 31 *Let v_r be a pitfall of some graph G. Then*

$$G \in G_C \Longleftrightarrow G \setminus \{v_r\} \in G_C \,,$$

where $G \setminus \{v_r\}$ results from G by removing all edges adjacent to v_r and vertex v_r from G.

Proof. If $G \setminus \{v_r\} \in G_R$ holds, the robber simply identifies any visit of the cop of the pitfall v_r by the dominating vertex v_c and makes use of a strategy in $G \in G_R \setminus \{v_r\}$.

If $G \setminus \{v_r\} \in G_C$ holds, the cop wants to extend its winning strategy to G. The cop simply identifies any visit of the robber of the pitfall v_r as a visit of the dominating vertex v_c and makes use of the same strategy. □

Now, we have a simple characterization of G_C.

Theorem 32 *The graph G is in G_C, if and only if the successive removement of pitfalls finally ends in a single vertex. The classification of a graph can be computed in polynomial time.*

Proof. Lemma 31 gives the key argument, as the classification does not change by removing pitfalls. This means that we either end up in a graph with no pitfalls for $G \in G_R$ or in a single vertex for $G \in G_C$.

Checking the existance of a pitfall can be done locally for any vertex and its neighborship. After computing the neigborship sets, we can check the pitfall property for a vertex in a brute-force manner in $O(n^2)$ time and for all vertices in $O(n^3)$ time for a graph with n vertices. At most n reduction steps can be done. □

Exercise 16 *Design an efficient algorithm for checking the pitfall property of a single vertex and/or for the graph.*

The above shrinking process answers the classification question algorithmically in polynomial time. On the other hand we would like to construct arbitrary examples of representatives of G_C. It can be shown that G_C is closed under the operations *product of two graphs* and *reduction of a graph.*

The *product* $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, G_2)$ is defined by vertex set $V_1 \times V_2$ and an edge set by the folllowing rules: (v_1, v_2) and (w_1, w_2) of $V_1 \times V_2$ build an edge if:

1. $v_1 = w_1$ and $(v_2, w_2) \in E_2$ or
2. $(v_1, w_1) \in E_1$ and $v_2 = w_2$ or
3. $(v_1, w_1) \in E_1$ and $(v_2, w_2) \in E_2$.

Lemma 33 *If $G_1, G_2 \in G_C$, then $G_1 \times G_2 \in G_C$*

Proof. If the cop has a winning strategy for G_1 that starts in v_1^s and catches the robber in v_1^e and G_2 that starts in v_2^s and catches the robber in v_2^e, the cop can start in (v_1^s, v_2^s) apply the strategies simultaneously and finally catches the robber in a vertex (v_1', v_2'). This strategy is obviously correct. □

For a graph G and its subgraph H, the *retraction* from G to H is a mapping $\varphi : V(G) \mapsto V(H)$ of the vertices of $V(G)$ of G to the vertices $V(H)$ of H as follows: $\varphi(H) = H$ for $(u, v) \in E$ we have $(\varphi(v), \varphi(u)) \in E(H)$. The graph H is a retract of G, if a retraction from G to H exists.

Note that $G \setminus \{v_r\}$ for a pitfall v_r is a retract of G.

Lemma 34 *If $G \in G_C$, and graph H is a retract of G, then $H \in G_C$.*

Proof. Assume that $H \in G_R$ holds and let φ be the mapping for a retraction from G to H. We would like to show $G \in G_R$. We extend the winning strategy for H to a winning strategy of G as follows: R remains in H and identifies the moves of C in G as moves in H. That is, if C moves from v to u in G, the robber indentifies this move as a move from $\varphi(u)$ to $\varphi(w)$ which exists in H by definition of φ. The robber always moves according to H and cannot be catched. □

Note that, the above lemmata do not rely on the fact that there is only one cop.

Theorem 35 *The class of graphs G in G_C is closed under the operations product and retraction.*

3.1.3 How many cops are required?

Obviously, any graph with a 4-cycle will not belong to G_C, therefore it makes sense to think about more than one cop. For a graph G the *cop-number*, $c(G)$ denotes the minimum number of cops required to guarantee that $G \in G_C$ holds.

A *vertex cover* of a graph G is a subset $V_c \subseteq V$ so that any vertex $u \in V \setminus V_c$ has a neighbor in V_c. Therefore the minimum vertex cover is an upper bound on $c(G)$. First, we show that $c(G)$ can be arbitrarily large for some graphs.

Theorem 36 *Let $G = (V, E)$ be a graph with minimum degree n that contains neither 3- nor 4-cycles. We conclude $c(G) \geq n$.*

Proof. Let us assume that $n-1$ cops are sufficient. If G does not have a vertex cover of size smaller than n, the $n-1$ cops located in the beginning at $c_1, \ldots, c_{n-1}$ cannot prevent the robber to choose a safe vertex. So the robber choose such a vertex, whose neighbors are not occupied by the cops. Since there are no 3- and 4-cycles, by the next move a single cop cannot threaten (occupy and/or be adjacent to) two neighbors of the robber in the next step. Therefore, there is still one safe neighbor for the robber after the next move of the cops.

It remains to show that a vertex cover of size $< n$ does not exist. Consider any vertex set $V = \{v_1, \ldots, v_{n-1}\}$ of G and a vertex $w \neq v_i$ for $i = 1, \ldots, n-1$. Note that $|V| \geq n$ holds, so w exists. Now consider the neighborhood, $N(w)$, of w. Let it consists of k vertices $v_1, \ldots, v_k$ from V and $l-k$ vertices $w_1, \ldots, w_{l-k}$ not in V. We have $l \geq n$, $k \leq n-1$ and $l-k \geq 1$. There are no $3-$ and $4-$cycles, so $N(w_i) \cap N(w_j)$ has to be $\{w\}$ for $i \neq j$. If the set V is a vertex cover for G, any $N(w_i)$ has to contain a different vertex from V. But none of the $N(w_i)$s can contain a vertex of $v_1, \ldots, v_k$, since this would give a 3-cycle with w. This means that we require $l-k$ different vertices from $v_{k+1}, \ldots, v_{n-1}$ an n vertices from V in total, a contradiction. □

We can construct regular graphs of arbitrary size, which fulfill the condition of Theorem 36. The following Theorem is given by construction.

Theorem 37 *For every n there exists a graph without 3- or 4-cylces with minimum degree n. So, for any n there is a graph with $c(G) \geq n$.*

Proof. For $n = 2$ the simple 5-cycle will work. Note that C_5 is 3-colorable, which means that we color the vertices such that no two colors are adjacent. Three colors are required and sufficient for C_5. Inductively, we construct a 3-colorable graph with degree exactly n for any vertex v of G and without 3- and 4-cylcels. The coloring is required for maintaining the cylcle condition by construction.

The inductuion base for $n = 2$ was shown above. Let us assume that the statement holds for n. We consider for copies G_0, G_1, G_2, G_3 of a corresponding graph G for n as depicted in Figure 3.3. We build new edges with respect to the coloring of the vertices so that any vertex obtains an additional edge; see Figure 3.3. From G_i to G_j any exact copies of two vertices of a single colors are connected. For example from G_1 to G_2 all identical copies of color 3 are connected, from G_2 to G_0 all identical copies of color 2 are connected and so on. There is a unique correspondance as shown in Figure 3.3.

Since there are no cycles of size 3 or 4 in G_0, G_1, G_2, G_3 and any two edges between G_i and G_j make use of identical copies of the same color there are at least two edges between them in G_i and G_j respectively. For the inductive step, we require a *new* 3-coloring, which will be attained by interchanging the colors for example color 3 by 1 and color 1 by 3 in G_1 and color 2 by 1 and color 1 by 2 in G_2 and so on. Thus we maintain 3-coloring in G_0, G_1, G_3, G_4 and also for the connections. □

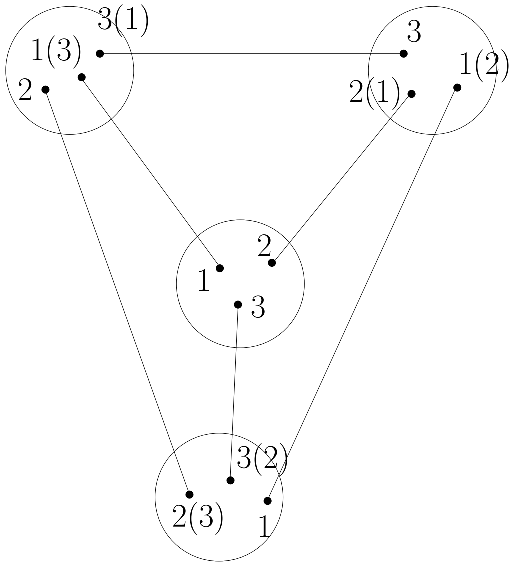

Figure 3.3: In the inductive step we use four copies G_0, G_1, G_2, G_3 of a 3-colorable graph G of degree exactly n without 3- and 4-cycles. Then we construct new edges according to the colors and finally interchange some colors, apropriately.

Finally, in this section we prove some positive results by bounding the cop-number from above for special graphs. The corresponding proofs are constructive, i.e., a winning strategy for the cops can be computed.

Theorem 38 *Consider a graph G with maximum degree 3 and the property that any two adjacent edges are contained in a cycle of lenght at most 5. Then $c(G) \leq 3$.*

Proof. The proof is constructive in the following sense. If the position of the robber is known, for the cops c_1, c_2 and c_3 we consider three paths toward r that uses all incident edges to r. We choose P_1, P_2 and P_3 for c_1, c_2 and c_3 respectively. The paths cover the incident edges by different cops and with length l_1, l_2 and l_3. And the paths make use of any possible shortcut for reaching the incident edges. Note that the paths need not be disjoint and r might also have only one or two incident vertices. But such paths do always exist. We would like to argue that by the condition of the Theorem, we can decrease the overall distance $l := l_1 + l_2 + l_3$ in any move of the cops.

Formally, after the move of the robber, R, we move c_1, c_2 and c_3 to c_1', c_2' and c_3' so that $l' < l$ holds. We further assume that r was adjacent to exactly three vertices r_1, r_2 and r_3. The other cases can be handled anlogously, and are given as an exercise. We have $P_1 = \{c_1, \ldots, r_1, r\}$, $P_2 = \{c_2, \ldots, r_2, r\}$ and $P_3 = \{c_3, \ldots, r_3, r\}$ and consider the following cases.

1. The robber R stands still. The cops move along the paths toward R and $l' \leq l - 3$.
2. The robber R moves to r_1 w.l.o.g.

   r_1 **has degree 1:** Either c_1 was on r_1 or $l_1 = 2$ and we are done or move all three cops toward r which gives $l' \leq l_1 - 2 + l_2 - 1 + l_3 - 1 = l - 4 < l$.

   r_1 **has degree 2:** Either c_1 was on r_1 and we are done or move all three cops toward r which gives $l' \leq l_1 - 2 + l_2 + l_3 = l - 2 < l$.

   r_1 **has degree 3:** Either c_1 was on r_1 and we are done or we have $l_1 \geq 2$. At least one adjacent vertex, say x, of r_1 does not belong to P_1, otherwise we use a shortcut for P_1.

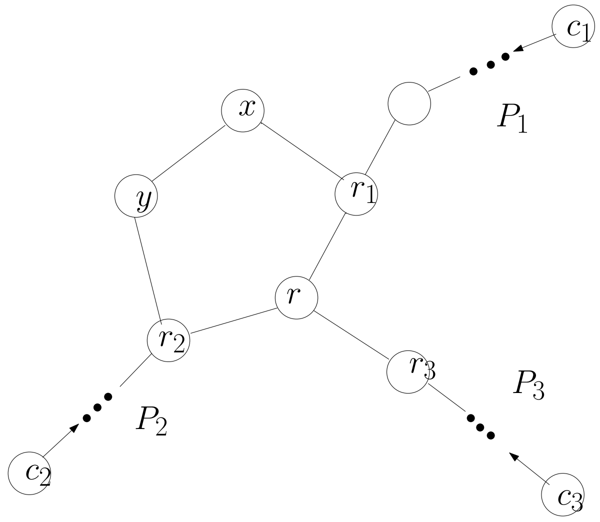

Figure 3.4: If r has degree 3 and c_1 is not on r_1, there is a 5-cycle so that we can move closer to r at least by one.

This means that (x, r_1) and (r_1, r) are on a cycle of length at most 5. Since r has only degree 3, one of the vertices r_2 or r_3, (say r_2) also belong to this cycle as depicted in Figure 3.4. So we have a 5-cycle r_2, y, x, r_1, r. We move all three cops toward r, respectively r_1 and use the paths $P_1 = \{c'_1, \ldots, r_1\}$ $P_2 = \{c'_2, \ldots, r_2, y, x, r_1\}$ and $P_3 = \{c'_3, \ldots, r_3, r, r_1\}$ with length $l' \leq l_1 - 2 + l_2 + 1 + l_3 = l - 1 < l'$.

In any case we can reduce the distance of the cops to the robber. □

Finally, we would like to prove that for any planar graph G, indeed $c(G) \leq 3$ holds. We first show that in any graph G it is always possible to protect a shortest path between two vertices by two cops. Protection means the robber cannot enter the path without beeing catched in the next step. The path lenght is given by the number of edges along a path between to vertices in G. By this measure the triangle inequality holds.

Lemma 39 *Consider a graph G and a shortest path $P = s, v_1, v_2, \ldots, v_n, t$ between two vertices s and t in G, assume that we have two cops. After a finite number of moves the path is protected by the cops so that after a visit of the robber R of a vertex of P the robber will be catched.*

Proof. First, we move a cop c onto some vertex $c = v_i$ of P. Let $d(x.y)$ denote the distance between two vertices in G. The robber r can only have a shorter distance to vertices on one side of P w.r.t. c because the triangle inequality holds. Assuming, that $r \neq v_i$ is closer to some x in $s, v_1, \ldots, v_{i-1}$ *and* some y in $v_{i+1}, \ldots, v_n, t$ is a contradiction to the shortest path between x and y. That is $d(x, c) + d(y, c) \leq d(x, r) + d(r, y)$. This means that $d(r, x) < d(c, x)$ only holds at most for one side of P w.r.t. c and for the other side we conclude $d(r, y) > d(c, y)$ in tis case.

Thus, we move c toward the vertices x. Now the robber can move. Again, if there are still vertices on one side of P w.r.t. c which are closer to r than to c we move further on toward these vertices. So finally, we achieved

$$d(r, v) \geq d(c, v) \text{ for all } v \in P \tag{3.1}$$

by this process.

Now the robber again could make its move. We show that we can also maintain the inequality all the time, which also means that the robber will be caught if he tries to move toward the vertices of P.

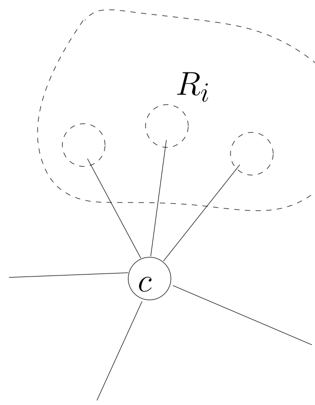

Figure 3.5: Case 1: All three cops in one vertex.

Assume Equation 3.1 holds. The robber can either stay in its place, so the cop c does and we fulfill Equation 3.1 (and the second cop could move now). Or the robber moves and tries to contradict Equation 3.1 by its single move. Assume r goes to some vertex r', we have

$$d(r', v) \geq d(r, v) - 1 \geq d(c, v) - 1 \text{ for all } v \in P .$$

If there is again some $v' \in P$ with $d(c, v') - 1 = d(r', v')$, we have the same situation as above and we can move c toward v' and Equation 3.1 holds again. Again as before the movement toward r' cannot reduce the distance to x and y on opposite sites of c w.r.t. P. Thus, by the move toward some v' we fulfill Equation 3.1. □

Finally, we exploit the above property for planar graphs and by the use of 3 cops and two such paths.

Theorem 40 *For any planar graph G we have $c(G) \leq 3$.*

Proof. We show that the region R_i for the robber R will shrink successively, that is $R_{i+1} \subset R_i$ after some moves of the cops. Two situations can appear.

Case 1: All three cops occupy a single vertex c and the robber is located in one component R_i of $G \setminus \{c\}$; see Figure **??**.

Case 2: There are two different paths P_1 and P_2 from v_1 to v_2 that are protected in the sense of Lemma 39 by cops c_1 and c_2; see Figure 3.6. In this case $P_1 \cup P_2$ subdivided G into an interior, I, and an exterior region E. That is $G \setminus (P_1 \cup P_2)$ has at least two components. W.l.o.g. we assume that R is located in the exterior $E = R_i$.

We will show that these two cases can appear successively and the region R_i of the robber will shrink. In the beginning all cops are located in a single vertex c and we are in case 1. We show how we handle the cases.

Movements in Case 1: We consider different situations for the neighbors of c:

c **has one neighbor in** R_i**:** Move all cops to this neighbor c' and consider $R_{i+1} = R_i \setminus \{c'\}$. This gives Case 1 again.

c **has more than one neighbor in** R_i**:** Let a and b be two of the neighbors and let $R(a, b)$ be a shortest path in R_i between a and b. One cop remains in c, another cop protects the path $R(a, b)$ by Lemma 39. Thus $P_1 = a, c, b$ and $P_2 = P(a, b)$. We are in Case 2 with $R_{i+1} \subset R_i$.

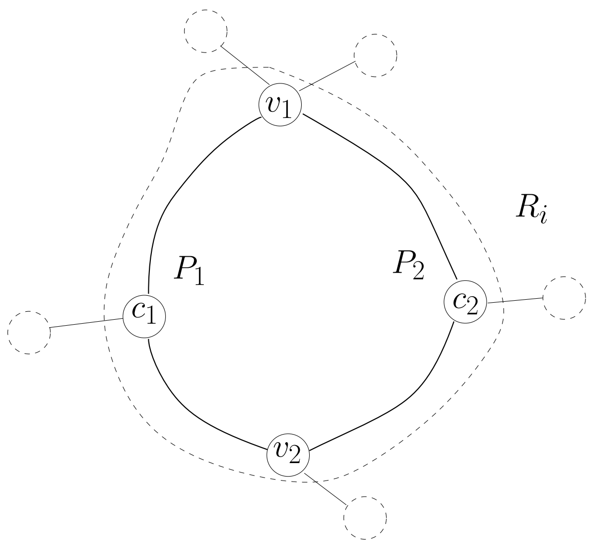

Figure 3.6: Case 2: Two cops protect two paths.

Movements for Case 2: We consider the situation for the composition of R_i and the location of the robber. We first assume that there is another shortest path different from P_1 and P_2 and partially running in R_i that connects v_1 and v_2. Let x_1 and x_2 be the first vertex where the path leaves and enters $P_1 \cup P_2$ respectively. We let c_3 protect the path P_3 which results from combining $P_{1,2}(v_1, x_1)$ with $x_1, r_1, \ldots, r_l, x_2$ combined with $P_{1,2}(x_2, v_2)$ as depicted in Figure 3.7. While c_1 and c_2 protect P_1 and P_2, the cop c_3 can protect this path. At the end c_3 protects P_3 and c_1 or c_2 the remaining path, we are in Case 2 with $R_{i+1} \subset R_i$.

On the other hand, if there is no path different from P_1 and P_2 and partially running in R_i that connects v_1 and v_2, there are no such leave and entry vertices x_1 and x_2 that are connected in R_i. Thus, the robber has to be inside a component that fully is connected to a single vertex x on P_1 and P_2. Thus we move c_3 to this vertex, and also c_1 and c_2 and end in Case 1 again. □

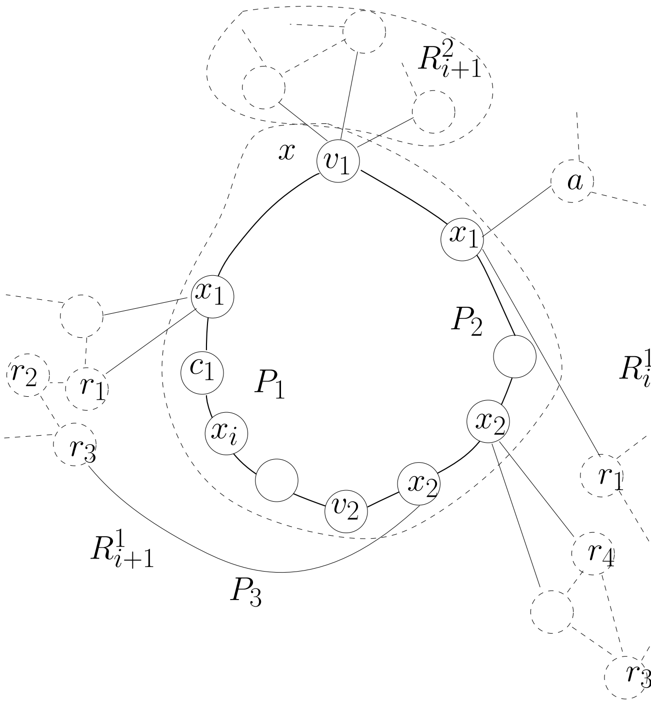

Figure 3.7: The situation for the two chains P_1 and P_2 protected by c_1 and c_2. If there is another shortest path from v_1 to v_2 different from P_1 and P_2 that runs partially in R_i, we construct a path P_3 that runs from v_1 to v_2 that is protected by c_3 alone and protects vertices of R_i. If there is no such path, a vetex x exists that can be visited by all cops and gives Case 1 again.