Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Cop and Robber Game

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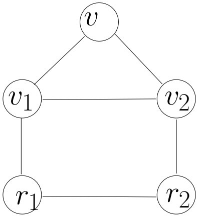
Cop and Robber Game in a graph

- Graph $G = (V, E)$
- Set the cop on a vertex
- Set the robber on a vertex
- Move alternatingly, try to visit robbers position

Cop and Robber game for graphs:
Instance: A Graph $G = (V, E)$ and the cardinality of the cops C.
Question: Is there a winning strategy S for the cops C?

Active and passive

Active version: Robbers *has to move* in each step!
Makes a difference!



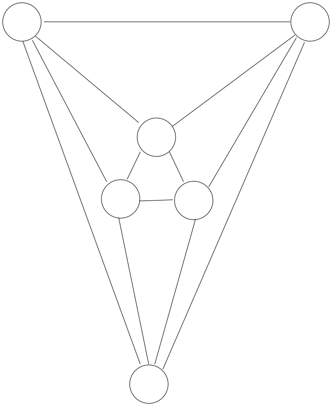
Classification and pitfalls

- Classes G_R and G_C for winning of cop or robber
- Situation at the end, single cop, G_C
- A pitfall for the robber
- Definitions

For a pair (v_r, v_c) of vertices we call v_r a *pitfall* and v_c its *dominating vertex* if $N(v_r) \cup \{v_r\} \subseteq N(v_c)$ holds. Obviously, a graph G whithout a pitfall is in G_R.

Graph without pitfalls

Graphs without pitfalls cannot have a winning strategy for the cop.



Successively, remove pitfalls is an algorithmic approach!

Lemma 31: Let v_r be a pitfall of some graph G. Then

$$G \in G_C \iff G \setminus \{v_r\} \in G_C$$

Proof:
1. $G \setminus \{v_r\} \in G_R \implies G \in G_R$ (pitfall by cop = dom vertex by cop)
2. $G \setminus \{v_r\} \in G_C \implies G \in G_C$ (pitfall by robber = dom. vertex by robber)

Algorithmis approach

Successively, remove pitfalls is an algorithmic approach!

Theorem 32: The graph G is in G_C, if and only if the successive removement of pitfalls finally ends in a single vertex. The classification of a graph can be computed in polynomial time.

Proof:
Lemma 31, remove a pitfall.
Detect a pitfall in polynomial time.
Example!

Arbitrary representatives

Product $G_1 \times G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, G_2)$
Vertex set $V_1 \times V_2$
Edges set: (v_1, v_2) and (w_1, w_2) of $V_1 \times V_2$ build an edge if:

1. $v_1 = w_1$ and $(v_2, w_2) \in E_2$ or
2. $(v_1, w_1) \in E_1$ and $v_2 = w_2$ or
3. $(v_1, w_1) \in E_1$ and $(v_2, w_2) \in E_2$.

Example!

Lemma 33: If $G_1, G_2 \in G_C$, then $G_1 \times G_2 \in G_C$

Proof:
Winning strategy for G_1 that starts in v_1^s and catches the robber in v_1^e and G_2 that starts in v_2^s and catches the robber in v_2^e.
Cop can start in (v_1^s, v_2^s) apply the strategies simultaneously and finally catches the robber in a vertex (v_1', v_2').

Arbitrary representatives, Retraction

- Graph G and subgraph H
- *Retraction* from G to H
- Mapping $\varphi : V(G) \mapsto V(H)$
- $\varphi(H) = H$ for $(u, v) \in E$ we have $(\varphi(v), \varphi(u)) \in E(H)$
- Graph H is a retract of G, if a retraction from G to H exists.

Note that $G \setminus \{v_r\}$ for a pitfall v_r is a retract of G. $\varphi(v_r) = v_c$.

Arbitrary representatives, Product

Lemma 34: If $G \in G_C$, and graph H is a retract of G, then $H \in G_C$.

Proof:

- Assume $H \in G_R$, φ mapping of retraction
- Winning strategy for H exists, extend to G
- R remains in H and identifies the moves of C in G as moves in H.
- C moves from v to u in G, the robber indentifies this move as a move from $\varphi(u)$ to $\varphi(w)$ which exists in H by definition of φ
- $G \in G_R$

Theorem 35: The class of graphs G in G_C is closed under the operations product and retraction.

Number of cops required

- Graph G with 4-cycle, one cop, $G \in G_R$
- $c(G)$, minimal number of cops required
- Vertex-Cover: $V_c \subseteq V$ so that any vertex $u \in V \setminus V_c$ has a neighbor in V_c.
- Minimum vertex cover is an upper bound on $c(G)$.
- $c(G)$ can be arbitraily large for some graphs

Number of cops required, negative results!

Theorem 36: Let $G = (V, E)$ be a graph with minimum degree n that contains neither 3- nor 4-cycles. We conclude $c(G) \geq n$.

Proof:

- Assume that $n - 1$ cops are sufficient
- Assume no vertex cover of size $< n$
- $c_1, \ldots, c_{n-1}$ starting positions
- Safe position for the robber, 2 steps away exists
- Next move of the cops
- No cop can threaten (occupy/be adjacent to) two neighbors of the robber, no such cycles
- Still one neigbor is safe!
- Show that no vertex cover of this size exists

Theorem 36: Let $G = (V, E)$ be a graph with minimum degree n that contains neither 3- nor 4-cycles. We conclude $c(G) \geq n$.

Proof:

- No vertex cover of size $n - 1$.
- Vertex set $V = \{v_1, \ldots, v_{n-1}\}$ of G
- $w \neq v_i$ for $i = 1, \ldots, n-1$ exists
- $N(w)$, of w: k vertices $v_1, \ldots, v_k$ from V and $l - k$ vertices $w_1, \ldots, w_{l-k}$ not in V
- We have $l \geq n$, $k \leq n - 1$ and $l - k \geq 1$
- No 3- and 4-cycles, $N(w_i) \cap N(w_j)$ has to be $\{w\}$ for $i \neq j$
- None of the $N(w_i)$s can contain a vertex of $v_1, \ldots, v_k$, since this would give a 3-cycle with w
- If the set V is a vertex cover for G, any $N(w_i)$ has to contain a different vertex from V.
- We require $l - k$ different vertices from $v_{k+1}, \ldots, v_{n-1}$ and n vertices from V in total, a contradiction.

Theorem 37: For every n there exists a graph without 3- or 4-cylces with minimum degree n. So, for any n there is a graph with $c(G) \geq n$.

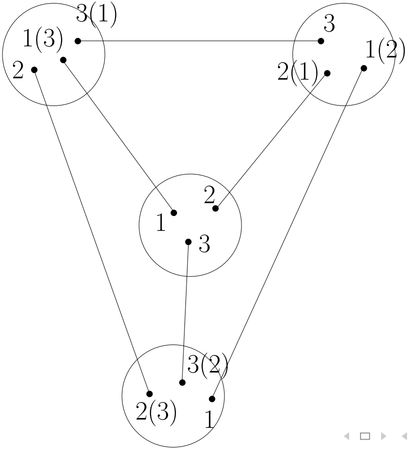
Proof:
By induction!

- $n = 2$ the simple 5-cycle
- 3-colorable and degree $\geq n$. At least n agents
- From n to $n + 1$!

Number of cops required, negative results

Theorem 37: For every n there exists a graph without 3- or 4-cylces with minimum degree n. So, for any n there is a graph with $c(G) \geq n$.

Proof: Inductive step! Four copies!



Number of cops required, positive result

Theorem 38: Consider a graph G with maximum degree 3 and the property that any two adjacent edges are contained in a cycle of lenght at most 5. Then $c(G) \leq 3$.

Proof:

- Position of the robber
- Build paths P_1, P_2 and P_3 from c_1, c_2, c_3 to adjacent edges
- Always move closer!
- $P_1 = \{c_1, \ldots, r_1, r\}$, $P_2 = \{c_2, \ldots, r_2, r\}$ and $P_3 = \{c_3, \ldots, r_3, r\}$
- $l = l_1 + l_2 + l_3$, decrease!

Number of cops required, positive result

Theorem 38: Consider a graph G with maximum degree 3 and the property that any two adjacent edges are contained in a cycle of lenght at most 5. Then $c(G) \leq 3$.

Proof:

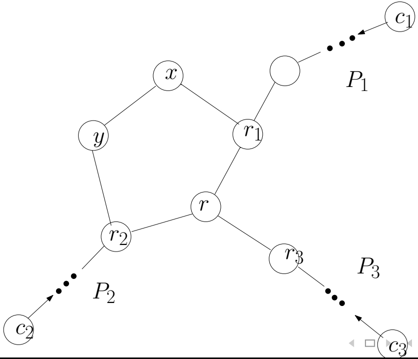
1. R stands still. Cops move toward R and $l' \leq l - 3$.
2. The robber R moves to r_1 w.l.o.g.

   r_1 has degree 1: Either c_1 was on r_1 or $l_1 = 2$ and we are done or move all three cops toward r which gives $l' \leq l_1 - 2 + l_2 - 1 + l_3 - 1 = l - 4 < l$.

   r_1 has degree 2: Either c_1 was on r_1 and we are done or move all three cops toward r which gives $l' \leq l_1 - 2 + l_2 + l_3 = l - 2 < l$.

Theorem 38: Consider a graph G with maximum degree 3 and the property that any two adjacent edges are contained in a cycle of lenght at most 5. Then $c(G) \leq 3$.

r_1 has degree 3: Situation as follows! Use the paths $P_1 = \{c_1', \ldots, r_1\}$ $P_2 = \{c_2', \ldots, r_2, y, x, r_1\}$ and $P_3 = \{c_3', \ldots, r_3, r, r_1\}$ with length $l' \leq l_1 - 2 + l_2 + 1 + l_3 = l - 1 < l'$.

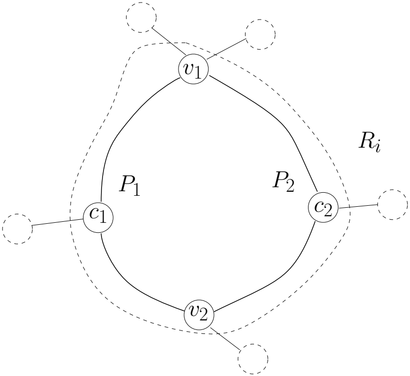


Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Proof:

- Two cops protect some paths, the third cop can proceed!



Number of cops required, positive result

Lemma 39: Consider a graph G and a shortest path $P = s, v_1, v_2, \ldots, v_n, t$ between two vertices s and t in G, assume that we have two cops. After a finite number of moves the path is protected by the cops so that after a visit of the robber R of a vertex of P the robber will be catched.

- Move cop c onto some vertex $c = v_i$ of P
- Assuming, $r \neq v_i$ closer to some x in $s, v_1, \ldots, v_{i-1}$ *and* some y in $v_{i+1}, \ldots, v_n, t$. Contradiction shortest path from x and y
- $d(x, c) + d(y, c) \leq d(x, r) + d(r, y)$
- Move toward x, finally: $d(r, v) \geq d(c, v)$ for all $v \in P$
- Now robot moves, but we can repair all the time
- r goes to some vertex r' and we have $d(r', v) \geq d(r, v) - 1 \geq d(c, v) - 1$ for all $v \in P$.
- Some $v' \in P$ with $d(c, v') - 1 = d(r', v')$ exists, move to v'

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Proof:

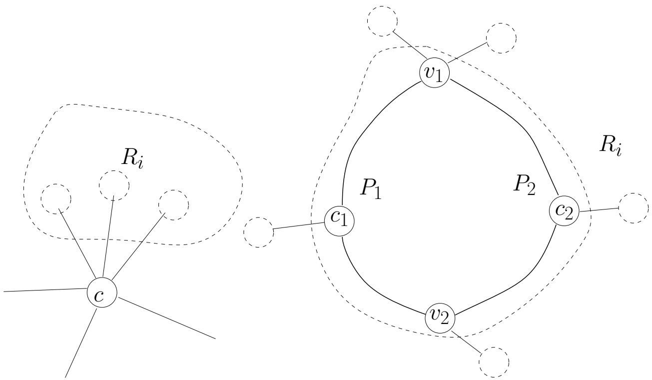
Case 1: All three cops occupy a single vertex c and the robber is located in one component R_i of $G \setminus \{c\}$

Case 2: There are two different paths P_1 and P_2 from v_1 to v_2 that are protected in the sense of Lemma 39 by cops c_1 and c_2. In this case $P_1 \cup P_2$ subdivided G into an interior, I, and an exterior region E. That is $G \setminus (P_1 \cup P_2)$ has at least two components. W.l.o.g. we assume that R is located in the exterior $E = R_i$.

Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 1 and Case 2



Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 1: Number of neighbors!

c neighbor in R_i: Move all cops to this neighbor c' and Consider $R_{i+1} = R_i \setminus \{c'\}$. Case 1 again.

c more than one neighbor in R_i: a and b be two neighbors, $R(a, b)$ a shortest path in R_i between a and b. One cop remains in c, another cop protects the path $R(a, b)$ by Lemma 39. Thus $P_1 = a, c, b$ and $P_2 = P(a, b)$. Case 2 with $R_{i+1} \subset R_i$.

Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 2: