Theoretical Aspects of Intruder Search

Course Wintersemester 2015/16
Cop and Robber Game Cont./Randomizations

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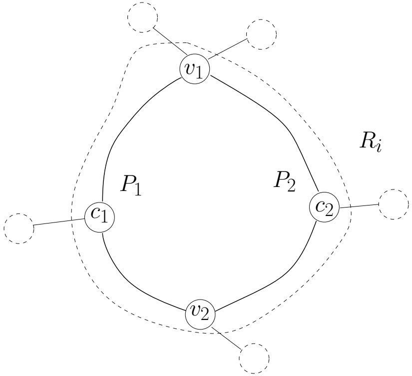
Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Proof:

- Two cops protect some paths, the third cop can proceed!



Lemma 39: Consider a graph G and a shortest path $P = s, v_1, v_2, \ldots, v_n, t$ between two vertices s and t in G, assume that we have two cops. After a finite number of moves the path is protected by the cops so that after a visit of the robber R of a vertex of P the robber will be catched.

- Move cop c onto some vertex $c = v_i$ of P
- Assuming, r closer to some x in $s, v_1, \ldots, v_{i-1}$ *and* some y in $v_{i+1}, \ldots, v_n, t$. Contradiction shortest path from x and y
- $d(x, c) + d(y, c) \leq d(x, r) + d(r, y)$
- Move toward x, finally: $d(r, v) \geq d(c, v)$ for all $v \in P$
- Now robot moves, but we can repair all the time
- r goes to some vertex r' and we have $d(r', v) \geq d(r, v) - 1 \geq d(c, v) - 1$ for all $v \in P$.
- Some $v' \in P$ with $d(c, v') - 1 = d(r', v')$ exists, move to v'

Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Proof:

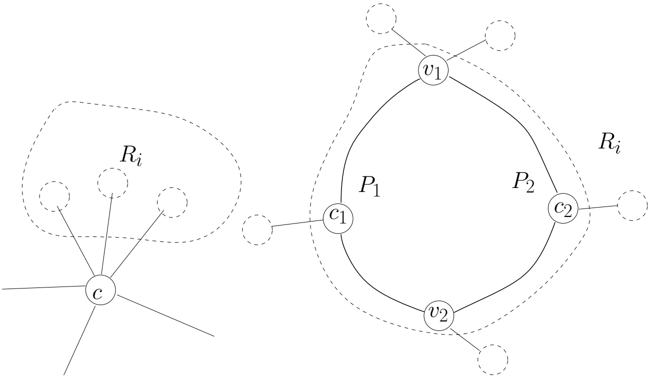
Case 1: All three cops occupy a single vertex c and the robber is located in one component R_i of $G \setminus \{c\}$

Case 2: There are two different paths P_1 and P_2 from v_1 to v_2 that are protected in the sense of Lemma 39 by cops c_1 and c_2. In this case $P_1 \cup P_2$ subdivided G into an interior, I, and an exterior region E. That is $G \setminus (P_1 \cup P_2)$ has at least two components. W.l.o.g. we assume that R is located in the exterior $E = R_i$.

Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 1 and Case 2



Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 1: Number of neighbors!

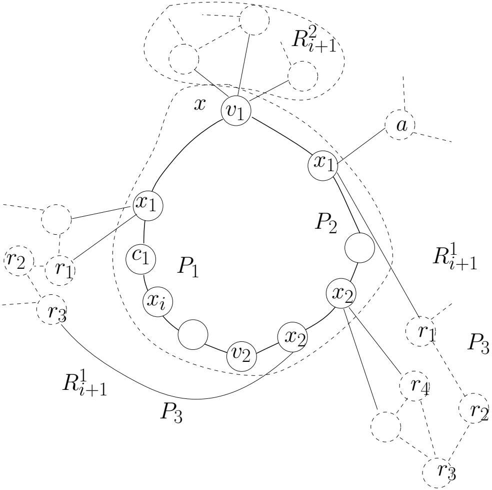
c one neighbor in R_i: Move all cops to this neighbor c' and Consider $R_{i+1} = R_i \setminus \{c'\}$. Case 1 again.

c more than one neighbor in R_i: a and b be two neighbors, $P(a, b)$ a shortest path in R_i between a and b. One cop remains in c, another cop protects the path $P(a, b)$ by Lemma 39. Thus $P_1 = a, c, b$ and $P_2 = P(a, b)$. Case 2 with $R_{i+1} \subset R_i$.

Number of cops required, positive result

Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 2:



Number of cops required, positive result

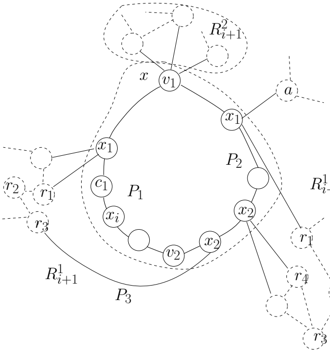
Theorem 40: For any planar graph G we have $c(G) \leq 3$.

Case 2:

1. There is a another shortest path $P'(v_1, v_2)$ in $P_1 \cup P_2 \cup R_i$ but different from P_1 and P_2. Leaves $P_1 \cup P_2$ at x_1, hits $P_1 \cup P_2$ again at x_2.
2. There is no such path! There is a single vertex x of $P_1 \cup P_2$ so that R is in the component *behind* x. Move all three cops to x. Case 1 again!

Number of cops required, positive result

Shortest path $P'(v_1, v_2)$ in $P_1 \cup P_2 \cup R_i$ but different from P_1 and P_2. Leaves $P_1 \cup P_2$ at x_1, hits $P_1 \cup P_2$ again at x_2.



Let c_3 protect $P_3 = v_1, \ldots, x_1, r_1, \ldots, r_k, x_2, \ldots, v_2$ while c_1 and c_2 protect $P_1 \cup P_2$.

Case 2 again: c_3 protects P_3, c_1 or c_2 the remaining one!

Aspects of randomization

- Examples for the use of randomizations
- Context of decontaminations
- Randomization for a strategy
- Beat the greedy algorithm for trees
- Randomization as part of the variant
- Probability distribution for the root
- Expected number of vertices saved

Beat the greedy approximation

Integer LP formlation for trees (Exercise):

$$
\begin{array}{llcll}
\text{Minimize} & \sum_{v \in V} x_v w_v & & & \\
\text{so that} & x_r = 0 & = & 0 & \\
& \sum_{v \leq u} x_v & \leq & 1 & : \quad \text{for every leaf } u \\
& \sum_{v \in L_i} x_v & \leq & 1 & : \quad \text{for every level } L_i, i \geq 1 \\
& x_v & \in & \{0,1\} & : \quad \forall\, v \in V
\end{array}
$$

Strategy: Beat the greedy approximation

- $\mathtt{opt}_{ILP}$ optimal solution, $\mathtt{opt}_{RLP}$ fractional solution, $\mathtt{opt}_{ILP} \leq \mathtt{opt}_{RLP}$
- $\mathtt{opt}_{RLP}$ in polynomial time!
- Subtree T_v with $x_v = a \leq 1$ is *a-saved*, a portion $a \cdot w_v$ of the subtree is saved
- v_1 is ancestor of v_2 and $x_{v_1} = a_1$ and $x_{v_2} = a_2$
- Vertices of T_{v_2} are $(a_1 + a_2)$-saved. The remaining vertices of T_{v_1} are only a_1-saved.
- Randomized rounding scheme for every level
- Sum of the $x_v = a$-values for level i: Probability distribution for choosing v. Shuffle and set x_v to 1.
- Sum up to less than 1: Probability of not choosing a vertex at level i.
- Only problem: *double-protections*

Strategy: Beat the greedy approximation

- *double-protections*: Choose vertices on the same path to a leaf! We only use the predecessor! Skip the higher level!
- No such *double-protections*: The expected approximation value would be indeed 1.
- Intuitive idea: Tree T_{v_i} at level i is *fully* saved by the fractional strategy!
- Worst-case: Fractional strategy has assigned a $1/i$ fraction to all vertices on the path from r to v_i. This gives 1 for T_{v_i}.
- Probability of saving v_i is: $1-(1-1/i)^i \geq 1-\frac{1}{e}$.
- Formal general proof!

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\mathtt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1-\frac{1}{e}\right)$.

- S_F fractional solution for $\mathtt{opt}_{RLP}$
- Probabilistic rounding scheme: S_I outcome of this assignment
- Show: Expected protection of S_I is larger than $\left(1-\frac{1}{e}\right)$ times the value of S_F
- x_v^F value of x_v for the fractional strategy
- x_v^I value $\{0,1\}$ of integer strategy
- $y_v = \sum_{u \leq v} x_u \in \{0,1\}$ indicate whether v is finally saved
- $y_v^F = \sum_{u \leq v} x_u^F \leq 1$ fraction of v saved by fractional strategy

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\texttt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

For $y_v = 1$ it suffices that one of the predecessor of v was chosen. Let $r = v_0, v_1, v_2, \ldots, v_k = v$ be the path from r to v

$$\mathbf{Pr}[y_v = 1] = 1 - \prod_{i=1}^{k} (1 - x_{v_i}^F) \,.$$

Explanation: The probability that v_2 is safe is
$x_1 + (1 - x_1)x_2 = 1 - (1 - x_1)(1 - x_2)$
The probability that v_3 is safe is
$1-(1-x_1)(1-x_2)+(1-x_1)(1-x_2)x_3 = 1-(1-x_1)(1-x_2)(1-x_3)$
and so on.

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\mathtt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

$$\begin{aligned}
\mathbf{Pr}[y_v = 1] &= 1 - \prod_{i=1}^{k}(1 - x_{v_i}^F) \\
&\geq 1 - \left(\frac{\sum_{i=1}^{k}(1 - x_{v_i}^F)}{k}\right)^k = 1 - \left(\frac{k - \sum_{i=1}^{k} x_{v_i}^F}{k}\right)^k \\
&= 1 - \left(\frac{k - y_v^F}{k}\right)^k \\
&= 1 - \left(1 - \frac{y_v^F}{k}\right)^k \geq 1 - e^{-y_v^F} \geq \left(1 - \frac{1}{e}\right) y_v^F .
\end{aligned}$$

$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}$

Approximation by randomized strategy

Theorem 41: Consider an algorithm that protects the vertices w.r.t. the probability distribution given by $\mathtt{opt}_{RLP}$. The expected approximation ratio of the above strategy for the number of vertices protected is $\left(1 - \frac{1}{e}\right)$.

$$\mathbf{E}(|S_I| = \sum_{v \in V} \mathbf{Pr}[y_v = 1] \geq \left(1 - \frac{1}{e}\right) \sum_{v \in V} y_v^F = \left(1 - \frac{1}{e}\right) |S_F| \,.$$

Randomization in variants of the problem

- $G = (V, E)$ fixed number k of agents
- k-surviving rate, $s_k(G)$, is the expectation of the *proportion* of vertices saved
- Any vertex is root vertex with the same probability
- Classes, C, of graphs G: For constant ϵ, $s_k(G) \geq \epsilon$
- Given G, k, $v \in V$ let:
  $\mathrm{sn}_k(G, v)$:number of vertices that can be protected by k agents, if the fire starts at v
- $\frac{1}{|V|} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \epsilon |V|$
- Class C: let the minimum number k that guarantees $s_k(G) > \epsilon$ for any $G \in C$ be denoted as the firefighter-number, $\mathrm{ffn}(C)$, of C.

Randomization in variants of the problem

Firefighter-Number for a class C of graphs:
Instance: A class C of graphs $G = (V, E)$.
Question: Assume that the fire breaks out at any vertex of a graph $G \in C$ with the same probability. Compute $\text{ffn}(C)$.

$\text{ffn}(C)$ for trees? For stars?

Planar graph: $\text{ffn}(C) \geq 2$, bipartite graph $K_{2,n-2}$.

Main Theorem: For planar graphs we have $2 \leq \text{ffn}(C) \leq 4$

Idea for the upper bound $\text{ffn}(C) \leq 4$

- Vertices subdivided into classes X and Y
- $r \in X$ allows to save many (a linear number of) vertices
- $r \in Y$ allows to save only few (almost zero) vertices
- Finally, $|Y| \leq c|X|$ gives the bound
- Simpler result first!

Simple proof!

Theorem : For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- Euler formula, $c + 1 = v - e + f$, for planar graphs, e edges, v vertices, f faces and c components
- Planar graph with no 3- and 4-cycle has average degree less than $\frac{10}{3}$
- Assume $\frac{10}{3}v \geq 2e$! Which is $v \geq \frac{3}{5}e$
- Also conclude $5f \leq 2e$.
- Insert, contradiction!
- Similar arguments: A graph with no 3-, 4 and 5-cylces has average degree less than 3!

Simple proof!

Theorem : For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

Subdivide the vertices V of G into groups w.r.t. the degree and the neighborship

- Let X_2 denote the vertices of degree ≤ 2.
- Let Y_4 denote the vertices of degree ≥ 4.
- Let X_3 denote the vertices of degree exactly 3 but with at least one neighbor of degree ≤ 3.
- Let Y_3 denote the vertices of degree exacly 3 but with all neighbors having degree > 3 (degree 3 vertices not in X_3).

Let x_2,x_3,y_3 and y_4 denote cardinality of the sets

Simple proof!

Theorem : For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- $|V| = n$, $x_2 + x_3 + y_3 + y_4 = n$
- $v \in X_2$: save $n - 2$ vertices
- $v \in X_3$: save $n - 2$ vertices
- For starting vertices in Y_3 and Y_4, we assume that we can save nothing!
- Show: $s_2(G) \cdot n = \frac{1}{n} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \epsilon \cdot n$

$$\frac{1}{n^2} \sum_{v \in V} \mathrm{sn}_k(G, v) \geq \frac{1}{n^2}(x_2 + x_3)(n - 2) = \frac{n-2}{n} \cdot \frac{x_2 + x_3}{x_2 + x_3 + y_3 + y_4}$$

Simple proof!

Theorem : For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- Fixed relation between $x_2 + x_3$ and $y_3 + y_4$
- First: Correspondance between Y_3 and Y_4
- $G_Y = (V_Y, E_Y)$: Edges of G with precisely one vertex in Y_3 and one vertex in Y_4
- $3y_3$ edges, at most $y_3 + y_4$ vertices, bipartite
- Cylce: Forth and back from Y_3 to Y_4
- No cycle of size 5!
- Average degree of vertices of G_Y is at most 3
- Counting $3(y_3 + y_4)$, counts at least any edge twice, so $3(y_3 + y_4) \geq 6y_3$
- $y_3 \leq y_4$

Simple proof!

Theorem : For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

- Fixed relation between $x_2 + x_3$ and $y_3 + y_4$, $y_3 \leq y_4$
- Counting $\frac{10}{3}(x_2 + x_3 + y_3 + y_4)$ edges we have at least counted $3x_3 + 3y_3 + 4y_4$ edges
- $9x_3 + 9y_3 + 12y_4 \leq 10(x_2 + x_3 + y_3 + y_4)$
- $2y_4 - y_3 \leq 10x_2 + x_3$
- By $y_3 \leq y_4$ we have $y_4 \leq 10x_2 + x_3$
- Finally: $y_3 + y_4 \leq 20x_2 + 2x_3 \leq 20(x_2 + x_3)$

Simple proof!

Theorem : For planar graphs G with no 3- and 4-cycle, we have $s_2(G) \geq 1/22$.

Finally: $y_3 + y_4 \leq 20x_2 + 2x_3 \leq 20(x_2 + x_3)$

$$\frac{n-2}{n} \cdot \frac{x_2 + x_3}{x_2 + x_3 + y_3 + y_4} \geq \frac{n-2}{n} \cdot \frac{x_2 + x_3}{21(x_2 + x_3)} = \frac{n-2}{21n}. \quad (1)$$

- $n = 2$: one vertex distinct from the root
- $3 \leq n \leq 44$: at least $\frac{2}{44}$
- $n \geq 44$: $s_2(G) \geq \frac{42}{21 \cdot 44} = \frac{1}{22}$.
- Expected value of saved vertices is always $\frac{1}{22}n$.

Warm up for planar graphs

Theorem 44: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

- Maximal, planar without multi-edges.
- Triangulation, any face has exactly 3 edges
- Subdivide V of G into sets X and Y.
- X will be the set of vertices strategy saves at least $n-6$ vertices
- Y we do not expect to save any vertex, for $|V| = n$
- Final conclusion is that for some $\alpha = \frac{1}{872}$

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| = 2709|X|. \tag{2}$$

Warm up for planar graphs

Theorem 44: Using four firefighters in the first step and then always three firefighters in each step, for every planar graph G there is a strategy such that $s_4(G) \geq \frac{1}{2712}$ holds.

$$|Y| \leq \left(93 + \frac{3}{\alpha}\right)|X| = 2709|X|\,. \tag{3}$$

Thus from $|X| + |Y| = n$ we conclude

$$s_4(G) \geq \frac{n-6}{n} \cdot \frac{|X|}{|X| + |Y|} > \frac{n-2}{n} \cdot \frac{|X|}{2710|X|} = \frac{n-6}{2710n}\,.$$

For $n \geq 10846$ we have

$$s_4(G) \geq \frac{1}{2710} - \frac{6}{4 \cdot 2710^2} \geq \frac{2710 - 3/2}{2710^2} \geq \frac{1}{2712}$$

For $2 \leq n < 10846$ we save at least $\min(4, n-1)$ in the first step, which gives also $s_4(G) \geq \frac{1}{2712}$.