

## Bayes-Nash Equilibria

*Instructor: Thomas Kesselheim*

We have spent the past weeks discussing dominant-strategy incentive compatible (truthful) mechanisms. In these mechanisms, for every agent it is always a dominant strategy to report the true value. A classic example is the second-price auction. Today, we will broaden our perspective: What statements can we make if the mechanism is not truthful? For example, if it is first-price auction?

A natural approach would be to consider Nash equilibria. For example, given tie breaking in our favor, the first-price auction has a pure Nash equilibria, in which everybody bids their value except for the bidder of highest value. She bids the second-highest value. The weakness of this approach is that it requires full information: Essentially, the bidders have to know the other values.

Today, we will get to know an equilibrium concept for *incomplete information*. The players know their own values but only have a *prior belief* about the other players' values.

### 1 Bayes-Nash Equilibria

We will assume that bidder  $i$ 's value  $v_i \in V_i$  is drawn independently from some distribution  $\mathcal{D}_i$ . These distributions are known to all bidders. A bidder chooses a bid  $b_i$  depending on the own valuation  $v_i$ , not knowing  $v_{-i}$  but only the distributions. We model this by saying that bidder  $i$  chooses a *bidding function*  $\beta_i: V_i \rightarrow B_i$ , mapping valuations to bids. Whenever the valuation is  $v_i$ , the bidder bids  $\beta_i(v_i)$ . For example, truthful bidding is represented by  $\beta_i(v_i) = v_i$ .

**Definition 16.1** (Bayes-Nash equilibrium). *A (pure) Bayes-Nash equilibrium (BNE) is a profile of bidding functions  $(\beta_i)_{i \in N}$ ,  $b_i: V_i \rightarrow B_i$  such that for all  $i \in N$ , all  $v_i \in V_i$ , and all  $b'_i \in B_i$*

$$\mathbf{E}_{v_{-i} \sim \mathcal{D}_{-i}} [u_i(\beta(v), v_i)] \geq \mathbf{E}_{v_{-i} \sim \mathcal{D}_{-i}} [u_i((b'_i, \beta_{-i}(v)), v_i)] \quad ,$$

where  $\beta(v) = (\beta_1(v_1), \dots, \beta_n(v_n))$ .

So, we take the perspective of a single bidder. She knows her own  $v_i$ . The other values  $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$  are drawn from  $\mathcal{D}_1, \dots, \mathcal{D}_{i-1}, \mathcal{D}_{i+1}, \dots, \mathcal{D}_n$  respectively. The bidding function now tells her to bid  $\beta_i(v_i)$ . In an equilibrium, no other bid should give a higher utility. The other bidders keep playing according to the respective bidding functions. This, in particular, means that no other bidding function yields a higher expected utility when also taking the expectation over  $v_i$ .

**Example 16.2.** *In a truthful mechanism,  $(\beta_i)_{i \in N}$  with  $\beta_i(v_i) = v_i$  for all  $i \in N$  and all  $v_i \in V_i$  is a Bayes-Nash equilibrium. It is not necessarily the only one.*

**Example 16.3.** *Consider a first-price auction with two bidders, in which  $\mathcal{D}_i$  is the uniform distribution on  $[0, 1]$ . Let us show that  $(\beta_i)_{i \in N}$  with  $\beta_i(v_i) = \frac{1}{2}v_i$  for all  $i \in N$  is a Bayes-Nash equilibrium.*

Observe that for symmetry reasons, it is enough to only consider bidder 1. Fix any  $v_1 \in V_1$  and let us write out the expected utility when bidding some arbitrary  $b'_1 \in B_1$ . The expectation is over bidder 2's value, respectively the bid.

$$\mathbf{E}_{v_2 \sim \mathcal{D}_2} [u_1((b'_1, \beta_2(v_2)), v_1)] = \int_0^1 u_1((b'_1, \beta_2(v_2)), v_1) dv_2 = \int_0^1 u_1\left(\left(b'_1, \frac{v_2}{2}\right), v_1\right) dv_2 \quad .$$

Here, we used that  $\beta_2(v_2) = \frac{v_2}{2}$ . Now, what is the value of  $u_1((b'_1, \frac{v_2}{2}), v_1)$ ? If  $b'_1 < \frac{v_2}{2}$ , then it is 0, if  $b'_1 > \frac{v_2}{2}$ , then it is  $v_1 - b'_1$ . Therefore

$$\mathbf{E}_{v_2 \sim \mathcal{D}_2} [u_1((b'_1, \beta_2(v_2)), v_1)] = \int_0^{2b'_1} (v_1 - b'_1) dv_2 + \int_{2b'_1}^1 0 dv_2 = 2b'_1(v_1 - b'_1) = \frac{v_1^2}{2} - 2\left(b'_1 - \frac{v_1}{2}\right)^2 .$$

We see that that the last term is maximized exactly for  $b'_1 = \frac{v_1}{2}$ , so for all  $v_1$  and  $b'_1$

$$\mathbf{E}_{v_2 \sim \mathcal{D}_2} \left[ u_1 \left( \left( \frac{v_1}{2}, \beta_2(v_2) \right), v_1 \right) \right] \geq \mathbf{E}_{v_2 \sim \mathcal{D}_2} [u_1((b'_1, \beta_2(v_2)), v_1)] ,$$

which is exactly the equilibrium condition.

## 2 Symmetric Bayes-Nash Equilibria of First-Price Auctions

We will derive a generalization of this equilibrium for arbitrary numbers of players  $n$  and arbitrary continuous, identical distributions  $\mathcal{D}_1, \dots, \mathcal{D}_n$ .

We will assume that for all  $i \in N$  and all  $x \in \mathbb{R}_{\geq 0}$

$$\Pr [v_i \leq x] = F(x) = \int_0^x f(t) dt .$$

We also write  $G(x)$  for  $(F(x))^{n-1}$ .

Let us assume that there is a Bayes-Nash equilibrium  $(\beta_i)_{i \in N}$  in which all functions are identical and differentiable. Then we have for all  $y \in \mathbb{R}_{\geq 0}$

$$\mathbf{E}_{v_{-i} \sim \mathcal{D}_{-i}} [u_i((y, \beta_{-i}(v)), v_i)] = (v_i - y) \Pr \left[ \bigwedge_{j \neq i} \beta_j(v_j) < b'_i \right] = (v_i - y) \prod_{j \neq i} \Pr [\beta_j(v_j) < y]$$

If we let  $\phi$  denote the inverse of  $\beta_i$ , then,  $\Pr [\beta_j(v_j) < y] = \Pr [v_j < \phi(y)] = F(\phi(y))$  as  $\beta_j = \beta_i$ . So we get

$$\mathbf{E}_{v_{-i} \sim \mathcal{D}_{-i}} [u_i((y, \beta_{-i}(v)), v_i)] = (v_i - y) \prod_{j \neq i} F(\phi(y)) = (v_i - y) G(\phi(y)) .$$

If  $\beta_i(v_i) = y$ , then  $y$  has to be a local maximum of the above function. That is

$$\frac{d}{dy} (v_i - y) G(\phi(y)) = 0 .$$

The derivative can be calculated by standard rules

$$\frac{d}{dy} (v_i - y) G(\phi(y)) = -G(\phi(y)) + (v_i - y) G'(\phi(y)) \phi'(y) .$$

By the inverse function theorem, we have  $\phi'(y) = \frac{1}{\beta'_i(\phi(y))}$ . That is, if  $\beta_i(v_i) = y$  then

$$-G(\phi(y)) + (v_i - y) G'(\phi(y)) \frac{1}{\beta'_i(\phi(y))} = 0 .$$

Replacing all occurrences of  $y$  by  $\beta_i(v_i)$  (so  $\phi(y) = v_i$ ), we get

$$-G(v_i) + (v_i - \beta_i(v_i)) G'(v_i) \frac{1}{\beta'_i(v_i)} = 0 ,$$

or equivalently

$$\beta'_i(v_i) G(v_i) + \beta_i(v_i) G'(v_i) = v_i G'(v_i) .$$

This has to hold for all  $v_i \in \mathbb{R}_{>0}$ . Observe that the left-hand side is exactly the derivative of  $\beta_i G$ . So, all solutions to this equation have the form

$$\beta_i(v_i)G(v_i) = \int v_i G'(v_i) dv_i + \text{constant} .$$

As  $\beta_i(0) = 0$ , we have

$$\beta_i(v_i) = \frac{1}{G(v_i)} \int_0^{v_i} t G'(t) dt .$$

One can verify that this is indeed an equilibrium the same way we did this in Example 16.3. And, as we have seen, it is necessarily the only symmetric equilibrium.

### 3 A Welfare Bound for First-Price Auctions

Let us have a closer look at the symmetric equilibrium that we have just derived. We observe that for any distribution the functions  $\beta_i$  are always strictly increasing. This means, whenever a bidder has a higher value, the bid will also be higher. Consequently, always the bidder with the highest value wins.

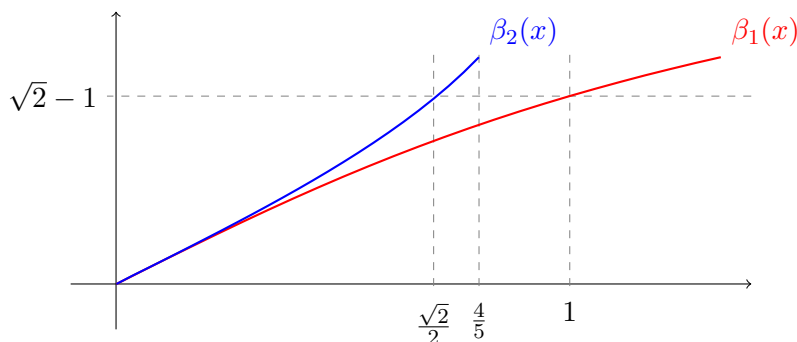
**Observation 16.4.** *In the symmetric Bayes-Nash equilibria  $(\beta_i)_{i \in N}$  of a first-price auction with identical distributions for all  $v \in V$*

$$\sum_{i \in N} v_i(f(\beta(v))) = \max_{i \in N} v_i .$$

If distributions are different, the equilibrium is usually asymmetric and it is not always true that the bidder with the highest value wins the item. For example, if  $v_1$  is uniformly distributed on  $[0, \frac{4}{3}]$  and  $v_2$  is uniformly distributed on  $[0, \frac{4}{5}]$ , then the unique Bayes-Nash equilibrium is

$$\beta_1(v_1) = -\frac{1 - \sqrt{1 + v_1^2}}{v_1} \quad \text{and} \quad \beta_2(v_2) = \frac{1 - \sqrt{1 - v_2^2}}{v_2} .$$

With constant probability, it happens that  $v_1 \in (\frac{4}{5}, 1]$  but  $v_2 \in (\frac{\sqrt{2}}{2}, \frac{4}{5}]$ . Whenever this is true,  $v_1 > v_2$  but  $\beta_1(v_1) \leq \beta_1(1) = \sqrt{2} - 1 = \beta_2(\frac{\sqrt{2}}{2}) < \beta_2(v_2)$ . So, bidder 2 wins despite having the smaller value.



However, we can still derive a guarantee. This is in the spirit of a Price-of-Anarchy bound.

**Theorem 16.5.** *In any Bayes-Nash equilibrium  $(\beta_i)_{i \in N}$  of a first-price auction*

$$\mathbf{E}_{v \sim \mathcal{D}} \left[ \sum_{i \in N} v_i(f(\beta(v))) \right] \geq \frac{1}{2} \mathbf{E}_{v \sim \mathcal{D}} \left[ \max_{i \in N} v_i \right] .$$

Before we come to the proof for Bayes-Nash equilibria, let us first see the argument in the full-information setting for pure Nash equilibria. That is, the valuations  $v$  and the bids  $b$  are fixed now.

It is important to observe that we can write the social welfare  $\sum_{i \in N} v_i(f(b))$  also as the sum of utilities and payments:  $\sum_{i \in N} v_i(f(b)) = \sum_{i \in N} u_i(b, v_i) + \sum_{i \in N} p_i(b)$ .

Let  $i^*$  be a player of maximum value. If this bidder now bids  $\frac{1}{2}v_{i^*}$ , then her utility is  $\frac{1}{2}v_{i^*}$  if she wins the item with this bid, meaning that  $\max_{i \neq i^*} b_i < \frac{1}{2}v_{i^*}$ . Otherwise it is 0. So, always the utility is at least  $\frac{1}{2}v_{i^*} - \max_{i \neq i^*} b_i$

As we are in an equilibrium,  $u_{i^*}(b, v_{i^*}) \geq u_{i^*}(\left(\frac{1}{2}v_{i^*}, b_{-i^*}\right), v_{i^*}) \geq \frac{1}{2}v_{i^*} - \max_i b_i$ . Also,  $u_i(b, v_i) \geq 0$  for all  $i \in N$  because one option would be  $b_i = 0$ . Therefore

$$\sum_{i \in N} u_i(b, v_i) + \sum_{i \in N} p_i(b) \geq \frac{1}{2}v_{i^*} - \max_i b_i + \sum_{i \in N} p_i(b) = \frac{1}{2}v_{i^*} .$$

*Proof.* We bound  $\mathbf{E}_{v \sim \mathcal{D}} [\sum_{i \in N} u_i(\beta(v), v_i)]$ . To this end, we use that for each bidder for each  $v_i$

$$\mathbf{E}_{v_{-i} \sim \mathcal{D}_{-i}} [u_i(\beta(v), v_i)] \geq \mathbf{E}_{v_{-i} \sim \mathcal{D}_{-i}} \left[ u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \right] .$$

This holds for every  $v_i$ , so it also holds if we draw  $v_i$  from  $\mathcal{D}_i$  and take this expectation:

$$\mathbf{E}_{v \sim \mathcal{D}} [u_i(\beta(v), v_i)] \geq \mathbf{E}_{v \sim \mathcal{D}} \left[ u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \right] .$$

And by linearity of expectation, we also get

$$\begin{aligned} \mathbf{E}_{v \sim \mathcal{D}} \left[ \sum_{i \in N} u_i(\beta(v), v_i) \right] &= \sum_{i \in N} \mathbf{E}_{v \sim \mathcal{D}} [u_i(\beta(v), v_i)] \\ &\geq \sum_{i \in N} \mathbf{E}_{v \sim \mathcal{D}} \left[ u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \right] \\ &= \mathbf{E}_{v \sim \mathcal{D}} \left[ \sum_{i \in N} u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \right] . \end{aligned}$$

For every fixed  $v$ , we also have

$$u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \geq \frac{v_i}{2} - \max_{i'} \beta_{i'}(v_{i'}) \quad \text{and} \quad u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \geq 0 .$$

This gives us

$$\sum_{i \in N} u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \geq \max_{i \in N} u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) \geq \max_{i \in N} \frac{v_i}{2} - \max_{i \in N} \beta_i(v) .$$

As we are in a first-price auction,  $\max_{i \in N} \beta_i(v) = \sum_{i \in N} p_i(\beta(v))$ , so

$$\sum_{i \in N} u_i \left( \left( \frac{v_i}{2}, \beta_{-i}(v) \right), v_i \right) + \sum_{i \in N} p_i(\beta(v)) \geq \max_{i \in N} \frac{v_i}{2} .$$

The rest follows directly by linearity of expectation. □

## 4 Outlook: Smooth Mechanisms

The last proof followed a very particular template: We use the fact that bidders do not want to deviate from the equilibrium to a fixed other strategy. We do not use further properties of the equilibrium—which is entirely different from the argument for symmetric equilibria. Indeed, there is a formalization of the latter proof pattern. In analogy to smooth games, we also call mechanisms smooth.

**Definition 16.6** (Smooth Mechanism, simplified version). *Let  $\lambda, \mu \geq 0$ . A mechanism  $M = (f, p)$ ,  $f: B \rightarrow X$ ,  $p: B \rightarrow \mathbb{R}^n$ , is  $(\lambda, \mu)$ -smooth if for any valuation profile  $v \in V$  for each player  $i \in \mathcal{N}$  there exists a bid  $b_i^*$  such that for any profile of bids  $b \in B$  we have*

$$\sum_{i \in \mathcal{N}} u_i(b_i^*, b_{-i}) \geq \lambda \cdot \max_{x \in X} \sum_{i \in \mathcal{N}} v_i(x) - \mu \sum_{i \in \mathcal{N}} p_i(b) .$$

In particular, our proof uses that a single-item first-price auction is  $(\frac{1}{2}, 1)$ -smooth. It uses  $b_i^* = \frac{v_i}{2}$ . Next time, we will once again see this definition and how it allows us to bound the welfare in equilibria of other mechanisms.